INSTITUTE OF PHYSICS PUBLISHING NONLINEARITY

Nonlinearity **16** (2003) 1607–1639 PII: S0951-7715(03)54436-X

Modulational stability of cellular flows

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Received 8 October 2002
Published 8 July 2003
Online at stacks.iop.org/Non/16/1607

Recommended by Weinan E

Abstract

We present here the homogenization of the equations for the initial modulational (large-scale) perturbations of stationary solutions of the two-dimensional Navier–Stokes equations with a time-independent periodic rapidly oscillating forcing. The stationary solutions are cellular flows and they are determined by the stream function $\phi = \sin x_1/\epsilon \sin x_2/\epsilon + \delta \cos x_1/\epsilon \cos x_2/\epsilon$, $0 \leqslant \delta \leqslant 1$. Two results are given here. For any Reynolds number we prove the homogenization of the linearized equations. For small Reynolds number we prove the homogenization for the fully nonlinear problem. These results show that the modulational stability of cellular flows is determined by the stability of the effective (homogenized) equations.

Mathematics Subject Classification: 76M50, 35B27, 74Q10

1. Introduction

The main physical reason for studying the problem of the modulational stability of cellular flows is to understand the phenomenon of eddy viscosity: in the presence of small-scale eddies the transport of large-scale vector quantities, e.g. mean velocity, can be accompanied by depleted, and, for large Reynolds number, even 'negative', diffusion (see, e.g., [45]). This is different from the transport of scalar quantities, e.g. mean temperature. In the latter case, the presence of microstructure only enhances diffusion. The concept of eddy viscosity is used in oceanography [31, 45, 25] (another application is in astrophysics [37, 27, 53]). Using homogenization, in this paper we show rigorously that modulational perturbations of periodic cellular flows converge to the solutions of the effective equation for a modulational perturbation as the period size $\epsilon \to 0$. Due to negative diffusion, at high Reynolds number the effective equation of large-scale transport becomes Hadamard ill-posed. In this case, we study linearized equations and regularize the problem by restricting the initial conditions to trigonometric polynomials.

Depleted eddy viscosity occurs already at small Reynolds number. In this case we study the full nonlinear problem and account for nonlinear effects by introducing spaces of functions of two variables $f(x, y)$, $y = x/\epsilon$ which Fourier transforms with respect to the first variable x has rapidly decaying Fourier coefficients.

1.1. Formulation

Suppose $\Phi^\epsilon(t, x)$ satisfies the incompressible Navier–Stokes equations in the vorticity-stream function form on $\Omega = \mathbb{T}^2 = [0, 2\pi] \times [0, 2\pi]$ driven by a time-independent $2\pi\epsilon$-periodic force $F(x/\epsilon)$, $x = (x_1, x_2)$

$$\begin{aligned} &\partial_t \Phi^\epsilon + \Delta^{-1} J_{xx}(\Phi^\epsilon, \Delta\Phi^\epsilon) = \nu\Delta\Phi^\epsilon + F\left(\frac{x}{\epsilon}\right), \\ &\Phi(t = 0, x) = \Phi_0\left(x, \frac{x}{\epsilon}\right), \end{aligned} \tag{1.1}$$

where $J_{xx}(u, v) = -\nabla_2 u \nabla_1 v + \nabla_1 u \nabla_2 v$. We assume that $\Phi^\epsilon(t, x)$ and $F(x/\epsilon)$ are periodic on Ω, therefore $\epsilon = 1/k$, $k \in \mathbb{Z}$. We also assume that $\Phi^\epsilon(t, x)$ and $F(x/\epsilon)$ are mean-zero. Then Δ^{-1} in (1.1) is the well-defined inverse of Laplacian on mean-zero periodic functions. We choose $F(x/\epsilon) = -\nu\Delta\phi(x/\epsilon)$, where $\phi(x/\epsilon)$ is the stream function of *cellular flows*:

$$\phi\left(\frac{x}{\epsilon}\right) = \sin\left(\frac{x_1}{\epsilon}\right)\sin\left(\frac{x_2}{\epsilon}\right) + \delta\cos\left(\frac{x_1}{\epsilon}\right)\cos\left(\frac{x_2}{\epsilon}\right), \qquad 0 \leqslant \delta \leqslant 1. \tag{1.2}$$

Cellular flows are a special class of periodic stationary solutions of the incompressible Euler equations ($\nu = 0$), that are used to model small-scale eddies—circular patterns in the fluid. When $\delta = 0$, they are also known as Taylor–Green flows (see, e.g., [41]).

The analysis presented in this paper applies to *any* flow, that satisfies

$$\Delta\phi = -\frac{\kappa}{\epsilon^2}\phi. \tag{1.3}$$

Condition (1.3) is a two-dimensional analogue of the Beltrami property (see, e.g., [4]), that implies that ϕ is a stationary solution of the incompressible Euler equations. The stream function (1.2) satisfies (1.3) with $\kappa = 2$. For simplicity of presentation, we concentrate here on cellular flows (1.2) only.

We study $\Phi^\epsilon(t, x)$, the solution of the Navier–Stokes equations (1.1), when it is given initially as a perturbation of the cellular flows (1.2): $\Phi^\epsilon(t = 0, x) = \phi + \Psi_0$. We are interested in linear and nonlinear stability of

$$\Psi^\epsilon(t, x) = \Phi^\epsilon(t, x) - \phi\left(\frac{x}{\epsilon}\right),$$

when the initial perturbation is *modulational*: $\Psi_0(x)$ is independent of a small parameter ϵ. $\Psi^\epsilon(t, x)$ satisfies the nonlinear modulation equation

$$\begin{aligned} &\partial_t \Psi^\epsilon(t, x) + \mathbb{N}^\epsilon(\Psi^\epsilon(t, x)) = \frac{1}{Re}\Delta\Psi^\epsilon(t, x) + M^\epsilon\Psi^\epsilon(t, x), \\ &\Psi^\epsilon(t = 0, x) = \Psi_0(x), \end{aligned} \tag{1.4}$$

where the Reynolds number, *Re*, is determined by the cellular flows: $Re = \|\phi\|_{L^\infty}/\nu$, the linear operator of *eddy viscosity* M^ϵ and the nonlinear term $\mathbb{N}^\epsilon$ are defined as

$$\begin{aligned} &M^\epsilon\Psi^\epsilon = -\Delta^{-1} J_{xx}\left(\phi\left(\frac{x}{\epsilon}\right), \left(\Delta + \frac{2}{\epsilon^2}\right)\Psi^\epsilon\right), \\ &\mathbb{N}^\epsilon(u, v) = \Delta^{-1} J_{xx}(u, \Delta v), \qquad \mathbb{N}^\epsilon(\Psi^\epsilon) = \mathbb{N}^\epsilon(\Psi^\epsilon, \Psi^\epsilon). \end{aligned} \tag{1.5}$$

If the nonlinear term in (1.4) can be neglected (e.g. when $\Psi^\epsilon(t,x)$ is sufficiently small), then $\Psi^\epsilon(t,x)$ satisfies the linearized modulation equation

$$\begin{aligned} &\partial_t \Psi^\epsilon(t,x) = \frac{1}{Re}\Delta\Psi^\epsilon + M^\epsilon \Psi^\epsilon(t,x), \\ &\Psi^\epsilon(t=0,x) = \Psi_0(x). \end{aligned} \tag{1.6}$$

For the analysis of eddy viscosity, the mathematical model (1.1) and (1.2) has been studied for a number of different stream functions ϕ, see e.g. [16, 20, 34, 42–44], a review on a related problem for inviscid flows can be found in [19]. Only for Kolmogorov shear flows $\phi = \cos(x_1/\epsilon)$, the one-dimensional nature of the problem allowed for a full modulational stability analysis including nonlinear effects (see [30, 51]).

Using a two-scale asymptotic expansion method (see, e.g., [7, 8, 21, 39]), it was shown in [35] that, if $\Psi^\epsilon(t,x) \to \Psi(t,x)$ as $\epsilon \to 0$, then $\Psi(t,x)$ satisfies the effective equation

$$\begin{aligned} &\partial_t \Psi(t,x) + \mathbb{N}(\Psi(t,x)) = \frac{1}{Re}\Delta\Psi(t,x) + M\Psi(t,x), \\ &\Psi(t=0,x) = \Psi_0(x), \end{aligned} \tag{1.7}$$

where

$$\begin{aligned} \mathbb{N}(\Psi) &= \mathbb{N}(\Psi,\Psi), \\ \mathbb{N}(u,v) &= \Delta^{-1} J_{xx}(u, \Delta v) \\ &\quad + \frac{Re^2}{8}\Delta^{-1}(\nabla_2^2 - \nabla_1^2)[\delta(\nabla_1 u \nabla_1 v + \nabla_2 u \nabla_2 v) + (1+\delta^2)\nabla_1 u \nabla_2 v], \end{aligned} \tag{1.8}$$

$$M\Psi(t,x) = -\frac{Re}{8}((\nabla_1 + \delta\nabla_2)^2 + (\delta\nabla_1 + \nabla_2)^2)\Psi + \left(\frac{Re}{2}(1+\delta^2) + \nu'\right)\Delta^{-1}(\nabla_2^2 - \nabla_1^2)^2\Psi. \tag{1.9}$$

The constant ν' in the equation for the averaged operator of eddy viscosity M satisfies $\nu' \geqslant 0$, $\nu' = \nu'(Re, \delta)$, and can be computed using the solution of a periodic boundary value (cell) problem (see equations (A.6) and (A.7) in appendix A). Asymptotics of ν' as $Re \to \infty$ is analysed in [35] by means of new saddle-point variational principles involving nonlocal operators.

In the linearized case the effective equation is

$$\begin{aligned} &\partial_t \Psi(t,x) = \frac{1}{Re}\Delta\Psi(t,x) + M\Psi(t,x), \\ &\Psi(t=0,x) = \Psi_0(x). \end{aligned} \tag{1.10}$$

The stability of the effective equations (1.7) and (1.10) was also studied in [35]. We assume their analysis and address the problem of stability of the modulation equations (1.4) and (1.6) by proving the convergence $\Psi^\epsilon(t,x) \to \Psi(t,x)$, as $\epsilon \to 0$.

1.2. Main results

We will employ the following notation throughout this paper. For any periodic mean-zero function $\Psi(t,x)$, its Fourier series representation is

$$\Psi(t,x) = \sum_{\boldsymbol{k}} \hat{\Psi}(t,\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\cdot x), \qquad \boldsymbol{k} \in \mathbb{Z}^2.$$

H^α is the Sobolev space of periodic mean-zero functions ($H^0 \equiv L_2$) with the norm, defined via the values of their Fourier coefficients:

$$\|\Psi(t,x)\|_{H^\alpha} = \left[\sum_{\boldsymbol{k}} |\boldsymbol{k}|^{2\alpha} |\hat{\Psi}(t,\boldsymbol{k})|^2\right]^{1/2}.$$

All the norms in this paper are with respect to the spatial variables only, hence they are functions of time. We will denote by C any positive constant whose value, which may change from line to line, is independent of any of the parameters. We write $C = C(\lambda_1, \lambda_2)$, if C depends on the parameters λ_1 and λ_2 only. Typically λ_1 and λ_2 are fixed, e.g. Reynolds number or a Sobolev norm of the initial conditions. When we need to track the value of a constant, we use bold face fonts and subscript indexes, e.g. $\boldsymbol{C}_0$. Our main results are presented in the following two theorems.

Theorem 1. *If $\Psi_0(x)$, the initial condition, is a finite trigonometric polynomial*

$$\Psi_0(x) = \sum_{|\boldsymbol{m}|<\boldsymbol{K}} a_{\boldsymbol{m}} \exp(\mathrm{i}\boldsymbol{m}\cdot x),$$

then, for any Re, all $t \in [0,\ T]$ and any $\beta > 0$, $\|\Psi^\epsilon(t,x) - \Psi(t,x)\|_{H^{1-\beta}} \to 0$, as $\epsilon \to 0$, where $\Psi^\epsilon(t,x)$ is the solution of the linearized modulation equation (1.6), $\Psi(t,x)$ is the solution of the effective linearized equation (1.10).

Theorem 2. *There exists a sufficiently small Reynolds number Re_0 such that for any $Re < Re_0$ there is a constant $\boldsymbol{C}_0 > 0$, $\boldsymbol{C}_0 = \boldsymbol{C}_0(Re)$ so that, if $\Psi_0(x)$, the initial condition, satisfies*

$$|\hat{\Psi}_0(\boldsymbol{m})| \leqslant \frac{\boldsymbol{C}_0}{|\boldsymbol{m}|^4}, \tag{1.11}$$

then, for any $t > 0$, $\Psi^\epsilon(t,x)$, the solution of the nonlinear modulation equation (1.4), satisfies

$$\|\Psi^\epsilon(t,x) - \Psi(t,x)\|_{H^{1-\beta}} \to 0, \qquad \text{as } \epsilon \to 0, \quad \text{for any } \beta > 0,$$

where $\Psi(t,x)$ is the solution of the effective nonlinear equation (1.7) which for all $t \geqslant 0$ satisfies

$$|\hat{\Psi}(t,\boldsymbol{m})| \leqslant \frac{\boldsymbol{C}_0}{|\boldsymbol{m}|^4}. \tag{1.12}$$

Some comments about the conditions in the theorems are in order. Numerical computation of the eigenvalues of the linearized modulation equation (1.6) shows that if the effective linearized equation (1.10) is stable, then the linearized modulation equation (1.6) is also stable not only with respect to modulational perturbations, but with respect to any initial perturbation. If the linearized modulation equation (1.6) is stable, then, using the method presented in this paper, it is fairly straightforward to prove stability of the nonlinear modulation equation (1.4). Therefore the critical Reynolds number in theorem 2 should be $Re_0 = 2\sqrt{2}/(1+\delta)$ (see linear stability analysis in [35]). In this paper we give a proof for an unspecified sufficiently small Re_0. Evaluation of the constants in our proof gives that $Re_0 \geqslant 0.01$. When $Re > 2\sqrt{2}/(1+\delta)$, the nonlinear modulation equation (1.4) is unstable (see [52]), but the question of the detailed behaviour of $\Psi^\epsilon(t,x)$, as $\epsilon \to 0$, remains open.

We show that there is a short time interval $[0, t_0]$, $t_0 = \mathrm{O}(\epsilon^2|\log\epsilon|) \to 0$, as $\epsilon \to 0$, where the small-scale part of $\Psi^\epsilon(t,x)$ changes rapidly. After t_0 the changes of the small-scale part of $\Psi^\epsilon(t,x)$ are not significant on such short time intervals. This means, loosely speaking, that in this short time period $\Psi^\epsilon(t,x)$ 'adjusts' to the presence of cellular flows. If initially $\Psi^\epsilon(t=0,x)$ is taken to be 'well-adjusted', then we can show that $\|\Psi^\epsilon(t,x) - \Psi(t,x)\|_{H^{1-\beta}} \leqslant \epsilon^\beta C$, $C = C(Re, \Psi_0(x))$. We do not present this better convergence estimate here.

The averaged operator of eddy viscosity M contains a negative diffusive term, which makes the problem (1.10) ill-posed for large Reynolds numbers: an initial condition with exponentially decaying spectrum is required if solutions are to exist for nonzero times. Such situations arise when the original (well-posed) problem is replaced by its leading order asymptotic model (see, e.g., [14]). It is of interest to determine a relevant formulation of the linearized modulation equation (1.6) so that the corresponding formulation of the effective linearized equation (1.10) makes the latter a well-posed problem. The proposed approach of assuming that the initial conditions are a trigonometric polynomial can be explained by the assumption of wide separation of scales: the starting point of the reduction of the full Navier–Stokes equations to a two-scale model is, in many cases, the assumption of the initial wide separation of scales (for our model see [16]). Mathematically, this assumption means that at $t = 0$ there is a gap in the Fourier spectrum between the small- and the large-scale Fourier modes. It implies that $\Psi_0(x)$, the large-scale part of the initial conditions of (1.1), has a compactly supported Fourier spectrum, or, equivalently for $\mathbb{T}^2$, finite number of Fourier coefficients.

The assumptions (1.11) and (1.12) are technical, and they arise from bilinear estimates on $\mathbb{N}(\Psi)$. The assumption (1.11), however, is not restrictive for our problem. Indeed, if $\Psi_0(x)$ is a modulational perturbation, it is a finite trigonometric polynomial, and its coefficients can be chosen arbitrarily small. Thus $\Psi_0(x)$ can be chosen to satisfy (1.11). Condition (1.12) comes from possible nonuniqueness of solutions of the effective nonlinear equation (1.7). The existence and uniqueness of solutions of the linearized equations (1.6) and (1.10) can be shown by standard energy methods. The existence and uniqueness of solutions of the nonlinear modulation equation (1.4) for any fixed ϵ follows from the existence and uniqueness results for the two-dimensional Navier–Stokes equations (see [12, 26, 48, 28]). But the existence and uniqueness of solutions of the effective nonlinear equation (1.7) is proved in this paper (theorem 3, section 3.1) only in the class (1.12). We are not interested in (possible) other solutions, because $\Psi^\epsilon(t, x)$, the solution of the nonlinear modulation equation (1.4), converges to $\Psi(t, x)$, the solution of the effective nonlinear equation (1.7), that satisfies (1.12). The estimates on $\mathbb{N}(\Psi)$, that lead to (1.12), are similar to classical estimates on the nonlinear term of the Navier–Stokes equations. For example, in [29] such estimates lead to a simple geometrical proof of the regularity the Navier–Stokes equations in two dimensions. As in the Navier–Stokes equations, the linear part of the nonlinear modulation equation (1.4) is diffusive for sufficiently small Reynolds numbers, but, in contrast to the Navier–Stokes equations, there are no energy-type estimates (e.g. enstrophy decay) to control the nonlinear term. We control it by choosing $\boldsymbol{C}_0$ sufficiently small and applying the contraction map argument.

The convergence results in theorems 1 and 2 basically say that $\Psi^\epsilon \to \Psi$ in a Sobolev space which is slightly smaller than H^1. The condition $\beta > 0$ is optimal: Ψ^ϵ does not converge to Ψ strongly in H^1 ($\beta = 0$). We show this for the linearized case (which implies the nonlinear case) while proving the homogenization (inequality (2.33) in section 2.4).

1.3. The method of proof

Let $\Psi^\epsilon(t, x)$ be a sequence of solutions of the modulation equations (1.4) or (1.6). To take the limit $\epsilon \to 0$ we need an *a priori* uniform in ϵ bound for $\Psi^\epsilon(t, x)$. In fact, provided a uniform bound is derived, the homogenization can be proved by classical techniques (e.g. the energy method [47, 32], the Γ-convergence method [15], the method of mesoscale characteristics [23, 9], the two-scale convergence method [1, 33]). Therefore we concentrate here on how we derive this *a priori* uniform bound. The key here is to be able to track *for all times* a certain symmetry condition of the small-scale part of $\Psi^\epsilon(t, x)$. If this condition is not satisfied, then the dynamics of the modulation equation is governed by the anisotropic kinetic alpha (AKA) effect (see [17, 46]), which is similar to the α-effect in magnetohydrodynamics (see [36] and references therein). If the AKA-effect is present, it requires to study, instead of a diffusive approximation (1.7), hyperbolic approximations as in [40] (see also [5, 6, 22] and references therein), because the timescale of the AKA-effect is 'much faster' than that of the eddy viscosity. To track the symmetry condition we use the dynamical systems approach (see, e.g., [13, 38, 49, 50]), that is we view the modulation equations (1.4) and (1.6) as an infinite-dimensional system of ordinary differential equations for the Fourier coefficients of solutions.

The detailed asymptotic behaviour of $\Psi^\epsilon(t, x)$ can be explained using the eigenvalue–eigenfunction (Bloch) decomposition [10, 18] (see also [11, 3, 2] and references therein) for the operator $\mathbb{L}^\epsilon = 1/Re\Delta + M^\epsilon$. Consider the linearized case. Then it is sufficient to study the linearized modulation equation (1.6) with initial conditions

$$\Psi_0(x) = \exp(\mathrm{i}\boldsymbol{m} \cdot x). \tag{1.13}$$

This leads to the shifted (Bloch) formulation

$$\frac{\partial}{\partial t}\Phi^\epsilon = \mathbb{L}^\epsilon(\boldsymbol{m})\Phi^\epsilon, \qquad \text{with } \Phi^\epsilon(t = 0, x) = 1,$$

where $\mathbb{L}^\epsilon(\boldsymbol{m}) = \exp(-\mathrm{i}\boldsymbol{m}x)\mathbb{L}^\epsilon \exp(\mathrm{i}\boldsymbol{m}x)$. $\mathbb{L}^\epsilon(\boldsymbol{m})$ is an operator with compact resolvent, therefore (see, e.g., [24]) there is a complete set of eigenfunctions and eigenvalues $\phi_i^\epsilon(\boldsymbol{m})$, $\lambda_i^\epsilon(\boldsymbol{m})$, $i = 0, 1, 2, \ldots$. The first eigenfunction of $\mathbb{L}^\epsilon(\boldsymbol{m})$ satisfies

$$\phi_0^\epsilon(\boldsymbol{m}) = 1 + \epsilon\xi(\boldsymbol{m}) + \epsilon^2\theta(\boldsymbol{m}) + \mathrm{O}(\epsilon), \qquad \text{as } \epsilon \to 0. \tag{1.14}$$

The symmetry of the Fourier coefficients of $\xi(\boldsymbol{m})$ (this is our symmetry condition) implies that the first eigenvalue $\lambda_0^\epsilon(\boldsymbol{m}) = \mathrm{O}(1)$, as $\epsilon \to 0$. All the other eigenvalues have negative real parts and they are $\mathrm{O}(1/\epsilon^2)$, as $\epsilon \to 0$. Therefore the solution of the linearized modulation equation (1.6) satisfies

$$\Psi^\epsilon = \exp(\lambda_0^\epsilon(\boldsymbol{m})t + \mathrm{i}\boldsymbol{m} \cdot x)(1 + \epsilon\xi(\boldsymbol{m}) + \epsilon^2\theta(\boldsymbol{m})) + \mathrm{O}(\epsilon^3), \qquad \text{as } \epsilon \to 0. \tag{1.15}$$

However, this last step of the explanation is difficult to justify analytically, because $\mathbb{L}^\epsilon(\boldsymbol{m})$ is a complicated operator, in particular, it is not normal (does not commute with its adjoint). Moreover, in general, (1.15) is false: the nonnormality $\mathbb{L}^\epsilon(\boldsymbol{m})$ may lead to growth of $\Psi^\epsilon(t, x)$. Therefore we use (1.14) and (1.15) only as intuition, when we construct the solution of the linearized modulation equation (1.6) as follows.

The Fourier coefficients of $\Psi^\epsilon(t, x)$, the solution of (1.6), satisfy an infinite-dimensional system of ordinary differential equations with coupling. The coupling comes from the operator of eddy viscosity M^ϵ, and it has a simple form: each Fourier coefficient $\hat{\Psi}^\epsilon(t, \boldsymbol{k})$ is completely determined, if we know four other Fourier coefficients $\hat{\Psi}^\epsilon(\tau, \bar{\boldsymbol{k}})$, $\bar{\boldsymbol{k}} = \boldsymbol{k} + (\pm 1, \pm 1)/\epsilon$, for $\tau \in [0, t]$, because

$$d_t\hat{\Psi}^\epsilon(t, \boldsymbol{k}) = -\frac{1}{Re}|\boldsymbol{k}|^2\hat{\Psi}^\epsilon(t, \boldsymbol{k}) + \sum_{\bar{\boldsymbol{k}}=\boldsymbol{k}+(\pm 1,\pm 1)/\epsilon} C(\bar{\boldsymbol{k}})\hat{\Psi}^\epsilon(t, \bar{\boldsymbol{k}}), \tag{1.16}$$

where $C(\bar{\boldsymbol{k}})$ are some constants, here and in the rest of this paper the notation $\pm\pm$, means all four pairs $(1, 1)$, $(-1, -1)$, $(1, -1)$ and $(-1, 1)$. Therefore, if Ψ^ϵ satisfies (1.13), then $\hat{\Psi}^\epsilon(t, \boldsymbol{k}) \equiv 0$, if $\boldsymbol{k} \neq \boldsymbol{m} + \boldsymbol{n}/\epsilon$, $\boldsymbol{n} \in \mathbb{Z}^2$. Then (1.6) can be identified with a problem on a lattice $\mathbb{Z}^2$, where at each vertex $\boldsymbol{n}$ there is a time-dependent function $\hat{\Psi}^\epsilon(t, \boldsymbol{m} + \boldsymbol{n}/\epsilon)$. Each function $\hat{\Psi}^\epsilon(t, \boldsymbol{m} + \boldsymbol{n}/\epsilon)$ satisfies an ordinary differential equation (1.16) with forcing, which is determined by functions $\hat{\Psi}^\epsilon(t, \boldsymbol{m} + \bar{\boldsymbol{n}}/\epsilon)$ at the 'nearest neighbours' of $\boldsymbol{n}$ on the lattice: $\bar{\boldsymbol{n}} = \boldsymbol{n} + (\pm 1, \pm 1)$. Assume that the large-scale part of Ψ^ϵ, denoted by

$$\Psi_l^\epsilon(t, x) = \hat{\Psi}^\epsilon(t, \boldsymbol{m}) \exp(\mathrm{i}\boldsymbol{m} \cdot x), \qquad \text{with } \hat{\Psi}^\epsilon(t = 0, \boldsymbol{m}) = 1$$

is given. Since the nearest neighbours of the origin are $\boldsymbol{n} = (\pm 1, \pm 1)$, $\Psi_l^\epsilon(t, x)$ 'forces' only the 'middle' small-scale part of $\Psi^\epsilon(t, x)$, denoted by

$$\Psi_m^\epsilon(t, x) = \sum_{\boldsymbol{n}=(\pm 1,\pm 1)} \hat{\Psi}^\epsilon \left(t, \boldsymbol{m} + \frac{\boldsymbol{n}}{\epsilon}\right) \exp\left(\mathrm{i}\left(\boldsymbol{m} + \frac{\boldsymbol{n}}{\epsilon}\right) \cdot x\right).$$

The forcing of each of the four Fourier coefficients of $\Psi_m^\epsilon(t, x)$ is determined by three 'high' small-scale Fourier coefficients of

$$\Psi_h^\epsilon(t, x) = \sum_{\boldsymbol{n}\neq(0,0),\boldsymbol{n}\neq(\pm 1,\pm 1)} \hat{\Psi}^\epsilon \left(t, \boldsymbol{m} + \frac{\boldsymbol{n}}{\epsilon}\right) \exp\left(\mathrm{i}\left(\boldsymbol{m} + \frac{\boldsymbol{n}}{\epsilon}\right) \cdot x\right)$$

and by the large-scale Fourier coefficient of $\Psi_l^\epsilon(t, x)$. By matching asymptotics, the forcing by $\Psi_l^\epsilon(t, x)$ is much stronger than that by $\Psi_h^\epsilon(t, x)$. Therefore we decompose

$$\Psi_m^\epsilon(t, x) = \Xi^\epsilon(t, x) + \Theta_m^\epsilon(t, x), \tag{1.17}$$

where the auxiliary function $\Xi^\epsilon(t, x)$ satisfies

$$\begin{aligned} &\partial_t \Xi^\epsilon(t, x) = \frac{1}{Re} \Delta \Xi^\epsilon(t, x) + M^\epsilon \Psi_l^\epsilon(t, x), \\ &\Xi^\epsilon(t = 0, x) = 0. \end{aligned} \tag{1.18}$$

The Fourier coefficient of $\Psi_l^\epsilon(t, x)$ is forced only by $\Psi_m^\epsilon(t, x)$:

$$d_t \hat{\Psi}^\epsilon(t, \boldsymbol{m}) = -\frac{1}{Re}|\boldsymbol{m}|^2 \hat{\Psi}^\epsilon(t, \boldsymbol{m}) + \frac{1}{\epsilon} F_1(t) + \frac{1}{\epsilon^2} F_2(t), \tag{1.19}$$

where $|F_1(t)| \leqslant C(\boldsymbol{m}) \|\Psi_m^\epsilon(t, x)\|_{L_2}$, and

$$\begin{aligned} |F_2(t)| \leqslant C\Bigg(&\left| \hat{\Psi}^\epsilon \left(t, \boldsymbol{m} + \frac{(1, 1)}{\epsilon}\right) + \hat{\Psi}^\epsilon \left(t, \boldsymbol{m} + \frac{(-1, -1)}{\epsilon}\right) \right| \\ &+ \left| \hat{\Psi}^\epsilon \left(t, \boldsymbol{m} + \frac{(-1, 1)}{\epsilon}\right) + \hat{\Psi}^\epsilon \left(t, \boldsymbol{m} + \frac{(1, -1)}{\epsilon}\right) \right| \Bigg). \end{aligned}$$

Therefore $\Psi_l^\epsilon(t, x)$ is bounded, as $\epsilon \to 0$, if $\|\Psi_m^\epsilon(t, x)\|_{L_2} = \mathrm{O}(\epsilon)$, as $\epsilon \to 0$, and if there are symmetries

$$\begin{aligned} &\left| \hat{\Psi}^\epsilon \left(t, \boldsymbol{m} + \frac{(1, 1)}{\epsilon}\right) + \hat{\Psi}^\epsilon \left(t, \boldsymbol{m} + \frac{(-1, -1)}{\epsilon}\right) \right| = \mathrm{O}(\epsilon^2), \\ &\left| \hat{\Psi}^\epsilon \left(t, \boldsymbol{m} + \frac{(-1, 1)}{\epsilon}\right) + \hat{\Psi}^\epsilon \left(t, \boldsymbol{m} + \frac{(1, -1)}{\epsilon}\right) \right| = \mathrm{O}(\epsilon^2) \end{aligned} \tag{1.20}$$

as $\epsilon \to 0$. Finally, denote $\Theta^\epsilon(t, x) = \Psi^\epsilon - \Psi_l^\epsilon - \Xi^\epsilon$. $\Theta^\epsilon(t, x)$ satisfies a partial differential equation with forcing, determined by $\Xi^\epsilon(t, x)$ only. It has the high small-scale part $\Theta_h^\epsilon(t, x) \equiv \Psi_h^\epsilon(t, x)$, and the middle small-scale part $\Theta_m^\epsilon(t, x)$ (cf (1.17)).

Our Ψ_l^ϵ, Ξ^ϵ, Θ^ϵ expansion is motivated by the Bloch decomposition. It is more complicated than (1.15), but it is more suitable for rigorous justification. We do not assume *a priori* that $\Xi^\epsilon = \mathrm{O}(\epsilon)$, $\Theta^\epsilon = \mathrm{O}(\epsilon^2)$, as $\epsilon \to 0$. We prove it by bootstrapping: suppose $\Psi_l^\epsilon(t, x)$ and $\Psi_m^\epsilon(t, x)$ are some functions, then $\Theta_h^\epsilon(t, x)$ is small, compared to $\Psi_m^\epsilon(t, x)$; then $\Theta_m^\epsilon(t, x)$ is small, compared to $\Xi^\epsilon(t, x)$; then $\Xi^\epsilon(t, x)$ is a so-called first order 'corrector' and we need to check (1.20) for $\Xi^\epsilon(t, x)$ only; then Ψ_l^ϵ must be bounded uniformly as $\epsilon \to 0$; then we improve estimates on $\Xi^\epsilon(t, x)$; then we improve estimates on $\Psi_h^\epsilon(t, x)$ and so on. Our bootstrapping proof relies on the following two observations. First, it is easy to track the symmetry condition for $\Xi^\epsilon(t, x)$, because it satisfies a simple heat equation (1.18). Second,

$\|\Psi_h^\epsilon(t,x)\|_{H^2}$ can be estimated by $\|\Psi_m^\epsilon(t,x)\|_{H^2}$ using classical energy estimates for parabolic equations. More specifically, for any χ

$$\int_\Omega \Delta\left(\Delta+\frac{2}{\epsilon^2}\right)\chi M^\epsilon\chi \equiv 0. \tag{1.21}$$

Therefore, if we multiply the linearized modulation equation (1.6) by $\Delta(\Delta+2/\epsilon^2)\Psi^\epsilon$, and integrate by parts

$$d_t E^\epsilon(t,\Psi^\epsilon) = -\frac{1}{Re}E_1^\epsilon(t,\Psi^\epsilon),$$

where

$$E^\epsilon(t,\Psi^\epsilon) = \|\Delta\Psi^\epsilon\|_{L_2}^2 - \frac{2}{\epsilon^2}\|\nabla\Psi^\epsilon\|_{L_2}^2, \qquad E_1^\epsilon(t,\Psi^\epsilon) = \|\nabla\Delta\Psi^\epsilon\|_{L_2}^2 - \frac{2}{\epsilon^2}\|\Delta\Psi^\epsilon\|_{L_2}^2.$$

In general, $E^\epsilon(t,\Psi^\epsilon)$ and $E_1^\epsilon(t,\Psi^\epsilon)$ are not equivalent to Sobolev norms—they are not positive-definite, but $E^\epsilon(t,\Psi_h^\epsilon)$ and $E_1^\epsilon(t,\Psi_h^\epsilon)$ are equivalent to H^2 and H^3 norms of $\Psi_h^\epsilon(t,x)$, respectively. Hence Gronwall's argument can be applied to derive *a priori* bounds on $\Psi_h^\epsilon(t,x)$.

For sufficiently small Re the linearized modulation equation (1.6) is stable, therefore the main issue in the nonlinear case is how to control the nonlinear term $\mathbb{N}^\epsilon(\Psi^\epsilon)$. View the solution of the nonlinear modulation equation (1.4) as a fixed point of a map $\mathbb{A}^\epsilon : \Psi^\epsilon \to \bar\Psi^\epsilon$ defined by

$$\partial_t\bar\Psi^\epsilon(t,x) = \frac{1}{Re}\Delta\bar\Psi^\epsilon(t,x) + [M^\epsilon\Psi^\epsilon + \mathbb{N}^\epsilon(\Psi^\epsilon)],$$

$$\bar\Psi^\epsilon(t=0,x) = \Psi_0(x).$$

Using estimates on the nonlinear term, we show that for sufficiently small Re and for all sufficiently small ϵ each $\mathbb{A}^\epsilon$ is a contraction on a ball of radius $\boldsymbol{C}_0 = \boldsymbol{C}_0(Re)$ in some Banach space L^ϵ. These Banach spaces are defined so that $\|\Psi^\epsilon\|_{L^\epsilon} \leqslant C\|\Psi^\epsilon\|_{H^1}$, which implies uniform boundedness of $\Psi^\epsilon(t,x)$ in $H^{1-\beta}$. The idea of the construction of L^ϵ is to assume that the Fourier coefficients of Ψ_l^ϵ satisfy the power decay (1.12), and then to follow the linearized case. Ψ_m^ϵ should have representation (1.17) where the term Ξ^ϵ should 'respect' the symmetry condition (1.20) by satisfying (1.18). For the Fourier coefficients of $\Psi_h^\epsilon \equiv \Theta_h^\epsilon$ and Θ_m^ϵ only bounds on their absolute values are required to be determined. For sufficiently small $Re < Re_0$, these bounds follow from the asymptotic analysis of [35]. It is convenient to set $\hat\Psi_l^\epsilon(t,\boldsymbol{m}) \equiv 0$, if $|\boldsymbol{m}| > \sigma/\epsilon$, $\sigma < \frac{1}{4}$, because this implies separation of scales of Ψ_l^ϵ, Ψ_m^ϵ and Ψ_h^ϵ: $\Psi_m^\epsilon(t,\boldsymbol{k}) \neq 0$ and $\Psi_h^\epsilon(t,\boldsymbol{k}) \neq 0$, when $\boldsymbol{k} = \boldsymbol{m} + \boldsymbol{n}/\epsilon$, $|\boldsymbol{m}| < \sigma/\epsilon$, with $\boldsymbol{n} = (\pm 1, \pm 1)$, and $|\boldsymbol{n}| > 2$, respectively. The Fourier coefficients $\Psi^\epsilon(t,\boldsymbol{k})$ when $\boldsymbol{k}$ lies outside the σ/ϵ 'neighbourhoods' of the vertices $\boldsymbol{n}/\epsilon$, $\boldsymbol{n} \in \mathbb{Z}^2$, determine the 'rest' of Ψ^ϵ:

$$\Psi_r^\epsilon = \Psi^\epsilon - [\Psi^\epsilon + \Psi_m^\epsilon + \Psi_h^\epsilon].$$

$|\Psi_r^\epsilon(t,\boldsymbol{k})|$ satisfies the power decay (1.12), hence Ψ_r^ϵ is negligible.

We call the spaces L^ϵ lacunary, because the absolute values of the Fourier coefficients of $\Psi^\epsilon \in L^\epsilon$, as a function on the lattice $\boldsymbol{k} \in \mathbb{Z}^2$, have 'bumps' in the neighbourhoods of $\boldsymbol{k} = \boldsymbol{n}/\epsilon$, $\boldsymbol{n} \in \mathbb{Z}^2$ and 'dips' away from these vertices.

The paper is organized as follows. There are two main parts: the proof of theorem 1 in section 2 and the proof of theorem 2 in section 3. Each of these proofs also has two parts: the proof of uniform boundedness of Ψ^ϵ (sections 2.3 and 3.3 for the linearized and nonlinear cases, respectively) and the proof of convergence, as $\epsilon \to 0$, (sections 2.4 and 3.4 for the linearized and nonlinear cases, respectively). In the linearized case, the proof of uniform boundedness of Ψ^ϵ relies on two *a priori* estimates: estimates on Ξ^ϵ given Ψ_l^ϵ (section 2.1), and estimates on Θ^ϵ given Ξ^ϵ (section 2.2). In the nonlinear case, the fixed point proof of uniform boundedness of Ψ^ϵ relies on the construction of lacunary spaces (section 3.2) and *a priori* estimates on the nonlinear term (appendix B). The proofs of convergence rely on the uniform boundedness of Ψ^ϵ, the same *a priori* estimates and estimates from the multiscale analysis of [35]. The latter can be found in appendix A. The proof of the stability theorem for the effective nonlinear equation (1.7) can be found in section 3.1.

2. Linearized case

2.1. Symmetries in the first-order corrector

Assume $\Psi_l^\epsilon(t,x) = a^\epsilon(t)\exp(\mathrm{i}\boldsymbol{m}\cdot x)$, then $\Xi^\epsilon(t,x)$ is a linear combination of four Fourier modes: $\Xi^\epsilon(t,x) = \sum a^\epsilon_{\pm\pm}(t)\exp(\mathrm{i}(\boldsymbol{m} + (\pm1,\pm1)/\epsilon)\cdot x)$. Denote

$$b^\epsilon(t) = \max_{0\leqslant\tau\leqslant t}|a^\epsilon(\tau)|, \qquad b_t^\epsilon(t) = \max_{0\leqslant\tau\leqslant t}|d_\tau a^\epsilon(\tau)|.$$

The main goal of this subsection is to give an explicit *a priori* estimate on the Fourier coefficients of Ξ^ϵ in terms of $b^\epsilon(t)$. These estimates will be used later to derive for any t the following bound

$$|\mathbb{P}^\epsilon_{\mathrm{large}} M^\epsilon\,\Xi^\epsilon|_{L_2} \leqslant C(Re)|\boldsymbol{m}|^2 b^\epsilon(t).$$

Lemma 1. *Suppose $\Xi^\epsilon(t,x)$ is the solution of (1.18). Then for any $t\in[0,T]$ and for $\boldsymbol{k} = \boldsymbol{m} + (\pm1,\pm1)/\epsilon$*

$$|a^\epsilon_{\pm\pm}(t)| \leqslant C\epsilon Re|\boldsymbol{m}|b^\epsilon(t) \leqslant CRe\frac{|\boldsymbol{m}|}{|\boldsymbol{k}|}b^\epsilon(t), \tag{2.1}$$

$$|a^\epsilon_{++}(t) + a^\epsilon_{--}(t)| \leqslant C\epsilon^2|\boldsymbol{m}|^2Re^2b^\epsilon(t) \leqslant CRe^2\frac{|\boldsymbol{m}|^2}{|\boldsymbol{k}|^2}b^\epsilon(t), \tag{2.2a}$$

$$|a^\epsilon_{+-}(t) + a^\epsilon_{-+}(t)| \leqslant C\epsilon^2|\boldsymbol{m}|^2Re^2b^\epsilon(t) \leqslant CRe^2\frac{|\boldsymbol{m}|^2}{|\boldsymbol{k}|^2}b^\epsilon(t). \tag{2.2b}$$

For any $t_0 > 0$ there is $\epsilon_0 = \epsilon_0(Re, t_0)$ such that for any $\epsilon \leqslant \epsilon_0$ and $t\in[t_0,T]$

$$|d_t a^\epsilon_{\pm\pm}(t)| = |\widehat{\partial_t\Xi^\epsilon}(t,\boldsymbol{k})| \leqslant C\epsilon Re|\boldsymbol{m}|b_t^\epsilon(t) \leqslant CRe\frac{|\boldsymbol{m}|}{|\boldsymbol{k}|}b_t^\epsilon(t), \tag{2.3}$$

$$\begin{aligned}&\Xi^\epsilon(t,x) = -Re\Delta^{-1}M^\epsilon\Psi_l^\epsilon(t,x) + \zeta_1^\epsilon(t,x),\\ &|\hat{\zeta}_1^\epsilon(t,\boldsymbol{k})| \leqslant CRe^2\epsilon^3|\boldsymbol{m}|b_t^\epsilon(t) \leqslant CRe^2\frac{|\boldsymbol{m}|}{|\boldsymbol{k}|^3}b_t^\epsilon(t).\end{aligned} \tag{2.4}$$

The proof of this lemma is technical, at it relies on the explicit evaluation of the heat kernel. The key bound that ensures the symmetry condition (2.2) is an estimate on the integral of the hyperbolic sine (see (2.7)).

Proof. It is sufficient to prove only the first inequalities in (2.1)–(2.4), because the second inequalities follow from $\epsilon \leqslant C/|\boldsymbol{k}|$, if $|\boldsymbol{m}| \leqslant \sigma/\epsilon$, and $\boldsymbol{k} = \boldsymbol{m} + (\pm1,\pm1)/\epsilon$.

Each Fourier coefficient $a^\epsilon_{\pm\pm}(t)$ satisfies an ordinary differential equation:

$$d_t a^\epsilon_{\pm\pm}(t) = \frac{1}{Re}\frac{1}{\epsilon^2}\Delta^\epsilon_{\pm\pm}a^\epsilon_{\pm\pm}(t) + \frac{1}{\epsilon}\boldsymbol{C}_{\pm\pm}a^\epsilon(t), \qquad a^\epsilon_{\pm\pm}(0) = 0, \tag{2.5}$$

where $\Delta^\epsilon_{\pm\pm} = -2 - \epsilon^2|\boldsymbol{m}|^2 - 2\epsilon(\pm m_1 \pm m_2)$, and the constants $\boldsymbol{C}_{\pm\pm}$ are explicitly given by

$$\boldsymbol{C}_{++} = -\frac{(m_1 - m_2)(1-\delta)(2-\epsilon^2|\boldsymbol{m}|^2)}{4((1+\epsilon m_1)^2 + (1+\epsilon m_2)^2)}, \qquad \boldsymbol{C}_{-+} = \frac{(m_1 + m_2)(1+\delta)(2-\epsilon^2|\boldsymbol{m}|^2)}{4((1-\epsilon m_1)^2 + (1+\epsilon m_2)^2)},$$

$$\boldsymbol{C}_{+-} = -\frac{(m_1 + m_2)(1+\delta)(2-\epsilon^2|\boldsymbol{m}|^2)}{4((1+\epsilon m_1)^2 + (1-\epsilon m_2)^2)}, \qquad \boldsymbol{C}_{--} = \frac{(m_1 - m_2)(1-\delta)(2-\epsilon^2|\boldsymbol{m}|^2)}{4((1-\epsilon m_1)^2 + (1-\epsilon m_2)^2)}.$$

Solving (2.5) we have

$$a^{\epsilon}_{\pm\pm}(t) = \epsilon \boldsymbol{C}_{\pm\pm} Re \int_0^{t/(\epsilon^2 Re)} \exp(\Delta^{\epsilon}_{\pm\pm}\tau) a^{\epsilon}(t - \epsilon^2 Re\,\tau)\,\mathrm{d}\tau. \tag{2.6}$$

Using $|\boldsymbol{C}_{\pm\pm}| \leqslant 1.5|\boldsymbol{m}|$ in (2.6) we have (2.1). From (2.6)

$$|a^{\epsilon}_{++}(t) + a^{\epsilon}_{--}(t)| \leqslant C\epsilon|\boldsymbol{m}|Re\,\mathbf{I},$$

where

$$\mathbf{I} = \left| \int_0^{t/(\epsilon^2 Re)} \exp(-(2+\epsilon^2|\boldsymbol{m}|^2)\tau)\sinh(\epsilon(m_1+m_2)\tau) a^{\epsilon}(t - \epsilon^2 Re\,\tau)\,\mathrm{d}\tau \right|$$

$$\leqslant C b^{\epsilon}(t) \frac{|m_1+m_2|\epsilon}{2+\epsilon^2|\boldsymbol{m}|^2 - (m_1+m_2)^2\epsilon^2} \leqslant C\epsilon|\boldsymbol{m}|b^{\epsilon}(t). \tag{2.7}$$

Therefore we have (2.2*a*). The derivation of (2.2*b*) is similar. Differentiating (2.5) we have that $\bar{a}^{\epsilon}_{\pm\pm}(t) = d_t a^{\epsilon}_{\pm\pm}(t)$ satisfies

$$d_t\bar{a}^{\epsilon}_{\pm\pm}(t) = \frac{1}{Re}\frac{1}{\epsilon^2}\Delta^{\epsilon}_{\pm\pm}\bar{a}^{\epsilon}_{\pm\pm}(t) + \frac{1}{\epsilon}\boldsymbol{C}_{\pm\pm}\bar{a}^{\epsilon}_{\pm\pm}(t), \qquad \bar{a}^{\epsilon}_{\pm\pm}(0) = \frac{\boldsymbol{C}_{\pm\pm}}{\epsilon}a^{\epsilon}(0).$$

Hence

$$|d_t a^{\epsilon}_{\pm\pm}(t)| \leqslant C\left[\exp\left(-\frac{t}{\epsilon^2 Re}\right)\frac{|\boldsymbol{m}|}{\epsilon}|a^{\epsilon}(0)| + \epsilon|\boldsymbol{m}|Re\, b^{\epsilon}_{\mathrm{t}}(t)\right].$$

For a fixed $t_0 > 0$ choose ϵ_0 such that $\exp(-t_0/(\epsilon_0^2 Re)) \leqslant \epsilon_0^2$, then by $|d_t a^{\epsilon}(0)| \leqslant |a^{\epsilon}(0)|/Re$ we have (2.3). Using (2.3) in (2.5) we have (2.4). ■

2.2. Energy estimates

Suppose $\Xi^{\epsilon} = \sum a^{\epsilon}_{\pm\pm}(t)\exp(\mathrm{i}(\boldsymbol{m} + (\pm1, \pm1)/\epsilon)\cdot x)$, then $\Theta^{\epsilon}(t,x) = \Psi^{\epsilon}_s - \Xi^{\epsilon}$ satisfies

$$\begin{aligned}
&\partial_t\Theta^{\epsilon}(t,x) = \frac{1}{Re}\Delta\Theta^{\epsilon}(t,x) + \mathbb{P}^{\epsilon}_{\mathrm{small}}M^{\epsilon}(\Theta^{\epsilon}(t,x) + \Xi^{\epsilon}(t,x)),\\
&\Theta^{\epsilon}(t=0,x) = 0.
\end{aligned} \tag{2.8}$$

The main goal of this subsection is to give an explicit *a priori* estimate on the Fourier coefficients of Θ^{ϵ}. These estimates will be used later to derive

$$|\mathbb{P}^{\epsilon}_{\mathrm{large}}M^{\epsilon}\Theta^{\epsilon}|_{L_2} \leqslant C(Re)|\boldsymbol{m}|^2 b^{\epsilon}(t)$$

for any t. Denote

$$\xi^{\epsilon}(t) = \max_{0\leqslant\tau\leqslant t}(\|\Xi^{\epsilon}(\tau,x)\|_{L_2}), \qquad \xi^{\epsilon}_{\mathrm{t}}(t) = \max_{0\leqslant\tau\leqslant t}(\|\partial_\tau\Xi^{\epsilon}(\tau,x)\|_{L_2}).$$

Lemma 2. *Suppose $\Theta^{\epsilon}(t,x)$ is the solution of (2.8). Then for any $t \in [0,T]$*

$$\|\Theta^{\epsilon}(t,x)\|_{H^s} \leqslant C\epsilon^{1-s}Re(Re+1)|\boldsymbol{m}|\xi^{\epsilon}(t), \qquad 0 \leqslant s \leqslant 2, \tag{2.9}$$

$$\|d_t\Theta^{\epsilon}(t,x)\|_{H^s} \leqslant C\epsilon^{1-s}Re(Re+1)|\boldsymbol{m}|\xi^{\epsilon}_t(t), \qquad 0 \leqslant s \leqslant 2, \tag{2.10}$$

$$\begin{aligned}
&\Theta^{\epsilon}(t,x) = -\left(\frac{1}{Re}\Delta + M^{\epsilon}\right)^{-1}\mathbb{P}^{\epsilon}_{\mathrm{small}}M^{\epsilon}\,\Xi^{\epsilon}(t,x) + \zeta^{\epsilon}_2(t,x),\\
&\|\zeta^{\epsilon}_2(t,x)\|_{H^s} \leqslant C\epsilon^{3-s}|\boldsymbol{m}|Re^2(Re+1)\xi^{\epsilon}_t(t), \qquad 0 \leqslant s \leqslant 2,
\end{aligned} \tag{2.11}$$

The proof of this lemma is standard, and it relies on energy inequalities and Gronwall's argument.

Proof. Recall $\Theta_h^\epsilon(t,x) = \mathbb{P}_{\text{high}}\Theta^\epsilon(t,x)$, and $\Theta_m^\epsilon(t,x) = \mathbb{P}_{\text{middle}}\Theta^\epsilon(t,x)$. Then $\Theta_m^\epsilon(t,x)$ is a linear combination of exactly four Fourier modes:

$$\Theta_m^\epsilon(t,x) = \sum \tilde{a}_{\pm\pm}^\epsilon(t)\exp\left(\mathrm{i}\left(\boldsymbol{m} + \frac{(\pm 1, \pm 1)}{\epsilon}\right)\cdot x\right).$$

Denote

$$\theta^\epsilon(t) = \max_{0\leqslant\tau\leqslant t}(\|\Theta_m^\epsilon(\tau,x)\|_{L_2}), \qquad \theta_t^\epsilon(t) = \max_{0\leqslant\tau\leqslant t}(\|\partial_\tau\Theta_m^\epsilon(\tau,x)\|_{L_2}).$$

$\Theta_h^\epsilon(t,x)$ satisfies

$$\partial_t\Theta_h^\epsilon(t,x) = \frac{1}{Re}\Delta\Theta_h^\epsilon(t,x) + \mathbb{P}_{\text{high}}^\epsilon M^\epsilon[\Xi^\epsilon(t,x) + \Theta_m^\epsilon(t,x) + \Theta_h^\epsilon(t,x)]. \tag{2.12}$$

Observe that if $\boldsymbol{m} \leqslant \sigma/\epsilon$, then

$$E^\epsilon(t) = \|\Delta\Theta_h^\epsilon(t,x)\|_{L_2}^2 - \frac{2}{\epsilon^2}\|\nabla\Theta_h^\epsilon(t,x)\|_{L_2}^2 > C\|\Theta_h^\epsilon(t,x)\|_{H^2}^2,$$

$$E_1^\epsilon(t) = \|\nabla\Delta\Theta_h^\epsilon(t,x)\|_{L_2}^2 - \frac{2}{\epsilon^2}\|\Delta\Theta_h^\epsilon(t,x)\|_{L_2}^2 > C\|\Theta_h^\epsilon(t,x)\|_{H^3}^2.$$

By Poincare's inequality $E^\epsilon(t) \leqslant E_1^\epsilon(t)/\epsilon^2$, and

$$\|\Theta_h^\epsilon(t,x)\|_{H^s}^2 \leqslant C\epsilon^{2(2-s)}E^\epsilon(t), \qquad 0\leqslant s\leqslant 2. \tag{2.13}$$

Assume that $\Theta_h^\epsilon(t,x) \in H^4$. Multiply (2.12) by $\Delta(\Delta + 2/\epsilon^2)\Theta_h^\epsilon(t,x)$ and integrate on the torus Ω. Then, using integration by parts, we have (recall (1.21)):

$$\partial_t E^\epsilon(t) = -\frac{1}{Re}E_1^\epsilon(t) + \mathbf{I} + \mathbf{II}, \tag{2.14}$$

where

$$\mathbf{I} = \int_\Omega \left(\Delta + \frac{2}{\epsilon^2}\right)\Theta_m^\epsilon(t,x)J_{xx}\left(\phi\left(\frac{x}{\epsilon}\right), \left(\Delta + \frac{2}{\epsilon^2}\right)\Theta_h^\epsilon(t,x)\right)\mathrm{d}\boldsymbol{x},$$

$$\mathbf{II} = \int_\Omega \left(\Delta + \frac{2}{\epsilon^2}\right)\Xi^\epsilon(t,x)J_{xx}\left(\phi\left(\frac{x}{\epsilon}\right), \left(\Delta + \frac{2}{\epsilon^2}\right)\Theta_h^\epsilon(t,x)\right)\mathrm{d}\boldsymbol{x}.$$

Therefore

$$\partial_t E^\epsilon(t) \leqslant -\frac{1}{Re}E_1^\epsilon(t) + |\mathbf{I}| + |\mathbf{II}|. \tag{2.15}$$

Since Θ_m^ϵ and Ξ^ϵ are just trigonometric polynomials for sufficiently small ϵ

$$\begin{aligned}\left\|\left(\Delta + \frac{2}{\epsilon^2}\right)\Theta_m^\epsilon(t,x)\right\|_{L_2} &\leqslant C\frac{|\boldsymbol{m}|}{\epsilon}\|\Theta_m^\epsilon(t,x)\|_{L_2} \leqslant C\frac{|\boldsymbol{m}|}{\epsilon}\theta^\epsilon(t),\\ \left\|\left(\Delta + \frac{2}{\epsilon^2}\right)\Xi^\epsilon(t,x)\right\|_{L_2} &\leqslant C\frac{|\boldsymbol{m}|}{\epsilon}\|\Xi^\epsilon(t,x)\|_{L_2} \leqslant C\frac{|\boldsymbol{m}|}{\epsilon}\xi^\epsilon(t).\end{aligned} \tag{2.16}$$

Also

$$\epsilon^2\left\|J_{xx}\left(\phi\left(\frac{x}{\epsilon}\right), \left(\Delta + \frac{2}{\epsilon^2}\right)\Theta_h^\epsilon(t,x)\right)\right\|_{L_2}^2 \leqslant \left\|\left(\Delta + \frac{2}{\epsilon^2}\right)\nabla\Theta_h^\epsilon(t,x)\right\|_{L_2}^2 \leqslant E_1^\epsilon(t).$$

Hence

$$|\mathbf{I}| \leqslant \frac{\epsilon^2}{4Re}\left\| J_{xx}\left(\phi\left(\frac{x}{\epsilon}\right), \left(\Delta + \frac{2}{\epsilon^2}\right)\Theta_h^\epsilon(t,x)\right)\right\|_{L_2} + \frac{2Re}{\epsilon^2}\left\|\left(\Delta + \frac{2}{\epsilon^2}\right)\Theta_m^\epsilon(t,x)\right\|_{L_2}^2 \leqslant \frac{1}{4Re}E_1^\epsilon(t) + C\frac{Re|\boldsymbol{m}|^2}{\epsilon^4}(\theta^\epsilon(t))^2, \tag{2.17}$$

$$|\mathbf{II}| \leqslant \frac{\epsilon^2}{4Re}\left\| J_{xx}\left(\phi\left(\frac{x}{\epsilon}\right), \left(\Delta + \frac{2}{\epsilon^2}\right)\Theta_h^\epsilon(t,x)\right)\right\|_{L_2} + \frac{2Re}{\epsilon^2}\left\|\left(\Delta + \frac{2}{\epsilon^2}\right)\Xi^\epsilon(t,x)\right\|_{L_2}^2 \leqslant \frac{1}{4Re}E_1^\epsilon(t) + C\frac{Re|\boldsymbol{m}|^2}{\epsilon^4}(\xi^\epsilon(t))^2. \tag{2.18}$$

Using (2.17) and (2.18) in (2.15) we have

$$d_t E^\epsilon(t) \leqslant -\frac{1}{2Re}E_1^\epsilon(t) + C\frac{Re|\boldsymbol{m}|^2}{\epsilon^4}[\xi^\epsilon(t) + \theta^\epsilon(t)]^2. \tag{2.19}$$

Since $E^\epsilon(t) \leqslant E_1^\epsilon(t)/\epsilon^2$, we have a differential inequality

$$d_t E^\epsilon(t) \leqslant -\frac{1}{2Re\,\epsilon^2}E^\epsilon(t) + C\frac{Re|\boldsymbol{m}|^2}{\epsilon^4}[\xi^\epsilon(t) + \theta^\epsilon(t)]^2.$$

Applying Gronwall's argument with $E^\epsilon(0) = 0$, we have

$$E^\epsilon(t) \leqslant CRe^2\frac{|\boldsymbol{m}|^2}{\epsilon^2}[\xi^\epsilon(t) + \theta^\epsilon(t)]^2.$$

Therefore by (2.13)

$$\|\Theta_h^\epsilon(t,x)\|_{H^s} \leqslant C\epsilon^{1-s}|\boldsymbol{m}|\,Re[\xi^\epsilon(t) + \theta^\epsilon(t)]. \tag{2.20}$$

Each Fourier coefficient $\tilde{a}_{\pm\pm}(t)$ of $\Theta_m^\epsilon(t,x)$ satisfies

$$d_t\tilde{a}_{\pm\pm}^\epsilon(t) = \frac{1}{Re}\frac{1}{\epsilon^2}\Delta_{\pm\pm}^\epsilon\tilde{a}_{\pm\pm}^\epsilon(t) + \frac{1}{\epsilon^2}G_{\pm\pm}^\epsilon(t), \tag{2.21}$$

where $|G_{\pm\pm}^\epsilon(t)| \leqslant C\epsilon^s\|\Theta_h^\epsilon(t,x)\|_{H^s}$, $0 \leqslant s \leqslant 2$ because each $G_{\pm\pm}^\epsilon(t)$ depends on the three Fourier coefficients $\hat{\Theta}_h^\epsilon(t,\boldsymbol{k})$ such that $\boldsymbol{k} = \boldsymbol{m} + \boldsymbol{n}/\epsilon$, $\boldsymbol{n} \neq (0,0)$, $\boldsymbol{n} = (\pm1,\pm1) + (\pm1,\pm1)$. Integrating (2.21) we have

$$\theta^\epsilon(t) \leqslant CRe\,\epsilon^s \max_{0\leqslant\tau\leqslant t}\|\Theta_h^\epsilon(\tau,x)\|_{H^s}, \qquad 0 \leqslant s \leqslant 2. \tag{2.22}$$

Inserting (2.22) in (2.20) we have

$$\max_{0\leqslant\tau\leqslant t}\|\Theta_h^\epsilon(\tau,x)\|_{H^s} \leqslant C\epsilon^{1-s}|\boldsymbol{m}|Re\,\xi^\epsilon(t), \qquad 0 \leqslant s \leqslant 2. \tag{2.23}$$

Combining (2.23) with (2.22) we have (2.9).

Differentiation of the first equation in (2.8) with respect to time gives that $\Theta_{dt}^\epsilon(t,x) = \partial_t\Theta^\epsilon(t,x)$ satisfies

$$\partial_t\Theta_{dt}^\epsilon(t,x) = \frac{1}{Re}\Delta\Theta_{dt}^\epsilon(t,x) + \mathbb{P}_{\text{small}}^\epsilon M^\epsilon(\Theta_{dt}^\epsilon(t,x) + \partial_t\Xi^\epsilon(t,x)),$$

$$\Theta_{dt}^\epsilon(t=0,x) = 0.$$

Then the same argument as for estimate (2.8) goes through, and hence (2.10) holds. Using (2.10) with (2.17), (2.18) and (2.13) in (2.14) we have (2.11).

By the classical Galerkin approximation methods the assumption $\Theta_h^\epsilon(t,x) \in H^4$ can be removed. If instead of (2.8) we consider a Galerkin approximation of the solution with finitely many Fourier modes then this approximation is a smooth function. Taking the limit when the number of the Galerkin modes approaches infinity we validate (2.9)–(2.11). ■

2.3. Uniform boundedness of solutions

By lemmas 1 and 2 in order to show uniform in ϵ boundedness of solutions of the linearized modulation equation (1.6) it is sufficient to show uniform boundedness of the large-scale part of these solutions.

Lemma 3. *If $\Psi_l^\epsilon(t,x)$, the large-scale part of the solution of the linearized modulation equation (1.6) with the initial condition $\Psi_0(x) = \exp(\mathrm{i}\boldsymbol{m}\cdot x)$, then there exists $\epsilon_0 = \sigma/|\boldsymbol{m}|$ such that for any Reynolds number Re, any $\epsilon < \epsilon_0$, and all $t \in [0,T]$, Ψ_l^ϵ satisfies*

$$\|\Psi_l^\epsilon(t,x)\|_{L_2} \leqslant \boldsymbol{C}_1(Re, |\boldsymbol{m}|, T), \qquad \|\partial_t \Psi_l^\epsilon(t,x)\|_{L_2} \leqslant \boldsymbol{C}_2(Re, |\boldsymbol{m}|, T). \tag{2.24}$$

The proof of this lemma relies on the explicit evaluation of the heat kernel. Using *a priori* estimates we derive that

$$|\mathbb{P}^\epsilon_{\text{large}} M^\epsilon(\Xi^\epsilon + \Theta^\epsilon)|_{L_2} \leqslant C(Re)|\boldsymbol{m}|^2 b^\epsilon(t)$$

for any t, which immediately leads to uniform in ϵ boundedness of Ψ_l^ϵ.

Proof. $a^\epsilon(t) = \hat{\Psi}^\epsilon(t,\boldsymbol{m})$ satisfies (cf (1.19)):

$$d_t a^\epsilon(t) = -\frac{1}{Re}|\boldsymbol{m}|^2 a^\epsilon(t) + B_1^\epsilon(t) + B_2^\epsilon(t) + G^\epsilon(t), \tag{2.25}$$

where

$$B_1^\epsilon(t) = \frac{1}{\epsilon}\left((a^\epsilon_{+-}(t) - a^\epsilon_{-+}(t))(m_1+m_2)\frac{1+\delta}{4} + (a^\epsilon_{++}(t) - a^\epsilon_{--}(t))(m_1-m_2)\frac{1-\delta}{4}\right),$$

$$B_2^\epsilon(t) = \frac{1}{\epsilon^2}\frac{m_2^2 - m_1^2}{2|\boldsymbol{m}|^2}[(a^\epsilon_{+-}(t) + a^\epsilon_{-+}(t))(1+\delta) + (a^\epsilon_{++}(t) + a^\epsilon_{--}(t))(1-\delta)],$$

$a^\epsilon_{\pm\pm}(t)$ are determined by $\Xi^\epsilon = \sum a^\epsilon_{\pm\pm}(t)\exp(\mathrm{i}(\boldsymbol{m} + (\pm 1, \pm 1)/\epsilon)\cdot x)$, and

$$|G^\epsilon(t)| \leqslant C\frac{1}{\epsilon^2}\|\Theta^\epsilon(t,x)\|_{L_2}. \tag{2.26}$$

We seek uniform in ϵ bounds on $B_1^\epsilon(t)$, $B_2^\epsilon(t)$ and $G^\epsilon(t)$ in terms of $b^\epsilon(t) = \max_{0\leqslant\tau\leqslant t}|a^\epsilon(\tau)|$. The crucial estimate is for $B_1^\epsilon(t)$. We use here (2.1):

$$|B_1^\epsilon(t)| \leqslant C|\boldsymbol{m}|^2 Re\, b^\epsilon(t). \tag{2.27}$$

For $B_2^\epsilon(t)$ we use (2.2*b*):

$$|B_2^\epsilon(t)| \leqslant C|\boldsymbol{m}|^2 Re^2 b^\epsilon(t). \tag{2.28}$$

For $G^\epsilon(t)$ we use lemma 2:

$$\|\Theta^\epsilon(t,x)\|_{L_2} \leqslant C\epsilon Re(Re+1)|\boldsymbol{m}|\xi^\epsilon(t), \tag{2.29}$$

therefore, combining (2.1), (2.26) and (2.29), we have

$$|G^\epsilon(t)| \leqslant CRe^2(Re+1)|\boldsymbol{m}|^2 b^\epsilon(t). \tag{2.30}$$

Using (2.27), (2.28) and (2.30) in (2.25) we have

$$\begin{aligned} d_t a^\epsilon(t) &= -\frac{1}{Re}|\boldsymbol{m}|^2 a^\epsilon(t) + B^\epsilon(t), \\ |B^\epsilon(t)| &\leqslant C|\boldsymbol{m}|^2 Re^2(Re+1) b^\epsilon(t). \end{aligned} \tag{2.31}$$

Therefore

$$|d_t b^\epsilon(t)| \leqslant C|\boldsymbol{m}|^2 Re^2(Re+1) b^\epsilon(t). \tag{2.32}$$

Hence $b^\epsilon(t) < C(Re, \boldsymbol{m}, T)$ for any t, $0 \leqslant t \leqslant T$ independent of ϵ. Using (2.31) with (2.32) we have (2.24) for any $\epsilon \leqslant \epsilon_0$ and $0 \leqslant t \leqslant T$. ■

2.4. Homogenization

Recall, that it is sufficient to prove theorem 1 with the initial condition $\Psi_0(x) = \exp(\mathrm{i}\boldsymbol{m}\cdot x)$. As $\epsilon \to 0$ by lemmas 1–3 for any fixed $t > 0$ the $H^{1-\beta}$-norm of $\Psi_s^\epsilon(t,x)$, the small-scale part of the solution of the linearized modulation equation (1.6), is negligible compared to $\|\Psi_l^\epsilon(t,x)\|_{H^{1-\beta}}$, if and only if $\beta > 0$:

$$\epsilon^\beta C_1 \leqslant \frac{\|\Psi_s^\epsilon(t,x)\|_{H^{1-\beta}}}{\|\Psi_l^\epsilon(t,x)\|_{H^{1-\beta}}} \leqslant \epsilon^\beta C_2, \tag{2.33}$$

where $C_1 = C_1(Re, \boldsymbol{m}, t) > 0$, $C_2 = C_2(Re, \boldsymbol{m}, t)$. Inequality (2.33) also implies that $\|\Psi^\epsilon(t,x) - \Psi(t,x)\|_{H^1} > C(Re, t, \boldsymbol{m})$ as $\epsilon \to 0$.

Since $\Psi_l^\epsilon(t,x) = \hat{\Psi}^\epsilon(t,\boldsymbol{m})\exp(\mathrm{i}\boldsymbol{m}\cdot x)$ and $\Psi(t,x) = \hat{\Psi}(t,\boldsymbol{m})\exp(\mathrm{i}\boldsymbol{m}\cdot x)$ we only need to show that

$$|\hat{\Psi}^\epsilon(t,\boldsymbol{m}) - \hat{\Psi}(t,\boldsymbol{m})| \leqslant \epsilon C(Re, \boldsymbol{m}, T) \qquad \text{for } 0 \leqslant t \leqslant T. \tag{2.34}$$

We prove (2.34) by showing that $\Psi_l^\epsilon(t,x)$ satisfies the effective linearized equation (1.10) with small forcing:

$$\begin{aligned} &\partial_t \Psi_l^\epsilon(t,x) = \frac{1}{Re}\Delta\Psi_l^\epsilon(t,x) + M\Psi_l^\epsilon(t,x) + F^\epsilon(t,x), \\ &\Psi_l^\epsilon(t=0,x) = \exp(\mathrm{i}\boldsymbol{m}\cdot x), \end{aligned} \tag{2.35}$$

where $F^\epsilon(t,x) = \hat{F}^\epsilon(t,\boldsymbol{m})\exp(\mathrm{i}\boldsymbol{m}\cdot x)$, and there exists $t_0^\epsilon \to 0$, as $\epsilon \to 0$, so that

$$|\hat{F}^\epsilon(t,\boldsymbol{m})| \leqslant \begin{cases} C(Re,\boldsymbol{m},T), & \text{if } 0 \leqslant t \leqslant t_0^\epsilon, \\ \epsilon C(Re,\boldsymbol{m},T), & \text{if } t_0^\epsilon \leqslant t \leqslant T. \end{cases} \tag{2.36}$$

By construction

$$F^\epsilon(t,x) = \mathbb{P}^\epsilon_{\text{large}} M^\epsilon(\Xi^\epsilon + \Theta^\epsilon) - M\Psi_l^\epsilon.$$

Utilizing estimates from lemmas 1 and 2 (specifically (2.4) and (2.11)), by lemma 5 of appendix A there exists $t_0^\epsilon \to 0$ as $\epsilon \to 0$ such that for $t \geqslant t_0^\epsilon$

$$|\hat{F}^\epsilon(t,\boldsymbol{m})| \leqslant C\epsilon|\boldsymbol{m}|^3\|\Psi_l^\epsilon(t,x)\|_{L_2}, \qquad C = C(Re).$$

Uniform boundedness of $\Psi_l^\epsilon(t,x)$ implies (2.35) with (2.36).

3. Nonlinear case. Small Reynolds number

3.1. Stability of the effective equation

Here we give a proof of the nonlinear stability of the effective nonlinear equation (1.7). It is worth mentioning that in this proof we use a nonlinear estimate for functions with rapidly decaying Fourier coefficients (see section 3.3) which is precisely the type of estimate we use later in the proof of homogenization of the nonlinear modulation equation.

Definition 1. $\Psi(t,x) \in L^0$ *if there is a constant* $\boldsymbol{C}_0 > 0$ *such that the Fourier coefficients of* $\Psi(t,x)$ *satisfy*

$$|\hat{\Psi}(\boldsymbol{m},t)| \leqslant \frac{\boldsymbol{C}_0}{|\boldsymbol{m}|^4}. \tag{3.1}$$

If $\Psi(t,x) \in L^\epsilon$, then the norm $\|\Psi(t,x)\|_{L^0} = \min(\boldsymbol{C}_0)$, where $\boldsymbol{C}_0$ are such that for them inequality (3.1) is satisfied.

Definition 2. $\mathbb{B}_{\boldsymbol{C}_0} = \{\Psi \in L^0 |$ *for any* $t > 0$, $\|\Psi(t,x)\|_{L^0} \leqslant \boldsymbol{C}_0\}$ *is a ball of radius* $\boldsymbol{C}_0$.

Theorem 3. *For any* $Re < Re_0 = 2\sqrt{2}/(1+\delta)$ *there is a constant* $\boldsymbol{C}_0 > 0$, $\boldsymbol{C}_0 = \boldsymbol{C}_0(Re)$ *such that for any time* $t > 0$ *there exists* $\Psi(t,x)$*, the unique solution of (1.7) with the initial conditions* $|\hat{\Psi}_0(\boldsymbol{m})| \leqslant \boldsymbol{C}_0/|\boldsymbol{m}|^4$*, that satisfies* $|\hat{\Psi}(t,\boldsymbol{m})| \leqslant \boldsymbol{C}_0/|\boldsymbol{m}|^4$ *for any* $t \geqslant 0$.

Proof. Consider a map $\mathbb{A} : \Psi \to \bar{\Psi}$, defined by

$$\begin{aligned} \partial_t \bar{\Psi}(t,x) &= \frac{1}{Re}\Delta\bar{\Psi}(t,x) + M\bar{\Psi}(t,x) - \mathbb{N}(\Psi(t,x)), \\ \bar{\Psi}(t=0,x) &= \Psi_0(x). \end{aligned} \tag{3.2}$$

We only need to show that for any $Re < Re_0 = 2\sqrt{2}/(1+\delta)$ there is a small constant $\boldsymbol{C}_0 > 0$ such that $\mathbb{A}$ is a contraction on $\mathbb{B}_{\boldsymbol{C}_0}$, that is if $\|\Psi_0(x)\|_{L^0} \leqslant \boldsymbol{C}_0$, then for any time $t > 0$, $\mathbb{A} : \mathbb{B}_{\boldsymbol{C}_0} \hookrightarrow \mathbb{B}_{\boldsymbol{C}_0}$, and

$$\|\mathbb{A}(\Psi_1(t,x) - \mathbb{A}(\Psi_2(t,x))\|_{L^0} \leqslant \tfrac{1}{2}\|\Psi_1(t,x) - \Psi_2(t,x)\|_{L^0}.$$

Suppose $\|\Psi\|_{L^0} \leqslant \boldsymbol{C}_0$. Using the Leibniz rule twice for $(\nabla_2^2 - \nabla_1^2)$ in (1.8), we have

$$|\widehat{\mathbb{N}(\Psi)}(t,\boldsymbol{m})| \leqslant C\frac{1}{|\boldsymbol{m}|^2}\sum_{\boldsymbol{m}=\boldsymbol{m}_1+\boldsymbol{m}_2}\left[\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_1|^3}\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_2|} + \frac{\boldsymbol{C}_0}{|\boldsymbol{m}_1|^2}\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_2|^2}\right] \leqslant \boldsymbol{C}_2\frac{\boldsymbol{C}_0^2}{|\boldsymbol{m}|^2}. \tag{3.3}$$

Observe that the eigenfunctions of the averaged operator of eddy viscosity are $\exp(\mathrm{i}\boldsymbol{m}\cdot x)$, $\boldsymbol{m} \in \mathbb{Z}^2$. If $Re < 2\sqrt{2}/(1+\delta)$, then the averaged operator of eddy viscosity is dominated by the Laplacian in the sense that for every Re there exists a constant $\mu = \mu(Re) > 0$ such that

$$\left[\frac{1}{Re}\Delta + M\right]\exp(\mathrm{i}\boldsymbol{m}\cdot x) = -\mu_{\boldsymbol{m}}\exp(\mathrm{i}\boldsymbol{m}\cdot x), \qquad |\boldsymbol{m}|^2\mu \leqslant \mu_{\boldsymbol{m}}.$$

Therefore, integrating (3.2) and using (3.3) and (3.1) we have

$$|\hat{\bar{\Psi}}(t,\boldsymbol{m})| \leqslant \frac{\boldsymbol{C}_0}{|\boldsymbol{m}|^4}\left[\exp(-\mu t) + (1-\exp(-\mu t))\frac{C\boldsymbol{C}_0}{\mu}\right].$$

Therefore, if $\boldsymbol{C}_0 \leqslant \mu/(2\boldsymbol{C}_2)$, and $\|\Psi(t,x)\|_{L^0} \leqslant \boldsymbol{C}_0$, then $\|\bar{\Psi}(t,x)\|_{L^0} \leqslant \boldsymbol{C}_0$.

If $\|\Psi_i(t,x)\|_{L^0} \leqslant \boldsymbol{C}_0$, $i = 1, 2$, $\|\Psi_1(t,x) - \Psi_2(t,x)\|_{L^0} \leqslant \boldsymbol{C}_1$, then

$$\begin{aligned} \|\mathbb{A}(\Psi_1) - \mathbb{A}(\Psi_2)\|_{L^0} &= \left\| \int_0^t \exp\left(\left(\frac{1}{Re}\Delta + M\right)\tau\right)\mathbb{N}(\Psi_1(t-\tau,x))\,\mathrm{d}\tau \right. \\ &\quad \left. - \int_0^t \exp\left(\left(\frac{1}{Re}\Delta + M\right)\tau\right)\mathbb{N}(\Psi_2(t-\tau,x))\,\mathrm{d}\tau \right\|_{L^0} \\ &\leqslant \boldsymbol{C}_1\frac{1}{2}(1-\exp(-\mu t)) \leqslant \frac{\boldsymbol{C}_1}{2}. \end{aligned}$$

■

3.2. Lacunary spaces

In this subsection we explicitly determine the Banach spaces of lacunary functions, that combine properties of functions with rapidly decaying Fourier coefficients and properties of two-scale solutions of the linearized modulation equation. When this goal is accomplished, we are able to prove homogenization of the fully nonlinear problem by combining the dynamical system technique of the linearized case and nonlinear energy estimates, similar to the one used in the proof of the stability of the effective nonlinear equation.

Identify with any $\boldsymbol{k} \in \mathbb{Z}^2$ a pair $(\boldsymbol{m}, \boldsymbol{n})$, $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{Z}^2$ chosen so that $\boldsymbol{m} = \boldsymbol{k} - \boldsymbol{n}/\epsilon$ has the minimal Euclidean norm $|\boldsymbol{m}| = \sqrt{m_1^2 + m_2^2}$ among all $\boldsymbol{n} \in \mathbb{Z}^2$ on the lattice $\boldsymbol{k} \in \mathbb{Z}^2$. Let us denote this as $\boldsymbol{k} = (\boldsymbol{m}, \boldsymbol{n})_\epsilon$ (when there is no confusion, we use this identification without mention). Then the σ/ϵ neighbourhoods (in Fourier space) of the vertices $\boldsymbol{k} = \boldsymbol{n}/\epsilon$, $\boldsymbol{n} \in \mathbb{Z}^2$ are determined by means of projection operators. The projection on the large scales is determined by

$$\mathbb{P}^\epsilon_{\text{large}} \exp(\mathrm{i}\boldsymbol{k}\cdot x) = \begin{cases} \exp(\mathrm{i}\boldsymbol{k}\cdot x), & \text{if } |\boldsymbol{k}| \leqslant \dfrac{\sigma}{\epsilon}, \\ 0, & \text{otherwise.} \end{cases}$$

Since the middle small scales play a distinguished role in our analysis—the 'nearest neighbours' of the large scales are located there, the projection operator on the small scales is decomposed into a sum of projections on middle and high small scales: $\mathbb{P}^\epsilon_{\text{small}} = \mathbb{P}^\epsilon_{\text{middle}} + \mathbb{P}^\epsilon_{\text{high}}$. The projection on the middle small scales is determined by

$$\mathbb{P}^\epsilon_{\text{middle}} \exp(\mathrm{i}\boldsymbol{k}\cdot x) = \begin{cases} \exp(\mathrm{i}\boldsymbol{k}\cdot x), & \text{if } \boldsymbol{n} = (\pm 1, \pm 1) \quad \text{and} \quad |\boldsymbol{m}| \leqslant \dfrac{\sigma}{\epsilon}, \\ 0, & \text{otherwise.} \end{cases}$$

The projection on the high small scales is determined by

$$\mathbb{P}^\epsilon_{\text{high}} \exp(\mathrm{i}\boldsymbol{k}\cdot x) = \begin{cases} \exp(\mathrm{i}\boldsymbol{k}\cdot x), & \text{if } |\boldsymbol{m}| \leqslant \dfrac{\sigma}{\epsilon}, \quad \|\boldsymbol{n}\| > 1, \\ 0, & \text{otherwise,} \end{cases}$$

where we denote $\|n\| = \max(|n_1|, |n_2|)$, for $\boldsymbol{n} = (n_1, n_2) \in \mathbb{Z}^2$. The σ/ϵ neighbourhoods are nonoverlapping, because $\sigma < \frac{1}{4}$. The projection on the 'rest' of the scales is determined by

$$\mathbb{P}^\epsilon_{\text{rest}} = \mathbf{1} - \mathbb{P}^\epsilon_{\text{large}} - \mathbb{P}^\epsilon_{\text{small}}.$$

Denote

$$\Psi^\epsilon_l = \mathbb{P}^\epsilon_{\text{large}}\Psi^\epsilon, \qquad \Psi^\epsilon_m = \mathbb{P}^\epsilon_{\text{middle}}\Psi^\epsilon, \qquad \Psi^\epsilon_h = \mathbb{P}^\epsilon_{\text{high}}\Psi^\epsilon, \qquad \Psi^\epsilon_r = \mathbb{P}^\epsilon_{\text{rest}}\Psi^\epsilon.$$

Definition 3. $\Psi^\epsilon(t, x) \in L^\epsilon$ *if there is a constant* $\boldsymbol{C}_0 > 0$ *such that* $\Psi^\epsilon(t, x)$ *can be decomposed into a lacunary sum*

$$\Psi^\epsilon(t, x) = \Psi^\epsilon_l(t, x) + \Xi^\epsilon(t, x) + \Theta^\epsilon(t, x) + \Psi^\epsilon_r(t, x), \tag{3.4}$$

where each term in (3.4) satisfies the following conditions. The function $\Psi^\epsilon_l(t, x)$ *satisfies*

$$|\hat{\Psi}^\epsilon_l(t, \boldsymbol{m})| \leqslant \frac{\boldsymbol{C}_0}{|\boldsymbol{m}|^4}, \qquad \mathbb{P}^\epsilon_{\text{small}}\Psi^\epsilon_l(t, x) \equiv 0, \qquad \mathbb{P}^\epsilon_{\text{rest}}\Psi^\epsilon_l(t, x) \equiv 0. \tag{3.5}$$

If $\breve{\Psi}^\epsilon_l(t, x)$ *is* another *function that satisfies (3.5), then* $\Xi^\epsilon(x, t)$ *is the solution of*

$$\begin{aligned} &\partial_t \Xi^\epsilon(t, x) = \frac{1}{Re}\Delta\Xi^\epsilon(t, x) + M^\epsilon\breve{\Psi}_l(t, x), \\ &\Xi^\epsilon(t = 0, x) = 0. \end{aligned} \tag{3.6}$$

The function $\Theta^\epsilon(t, x)$ *satisfies*

$$|\hat{\Theta}^\epsilon(t, \boldsymbol{k})| \leqslant \boldsymbol{C}_0\left(\frac{1}{2^{\|\boldsymbol{n}\|}|\boldsymbol{k}|^2|\boldsymbol{m}|^2} + \frac{1}{|\boldsymbol{k}|^3|\boldsymbol{m}|}\right), \qquad \mathbb{P}^\epsilon_{\text{large}}\Theta^\epsilon \equiv 0, \qquad \mathbb{P}^\epsilon_{\text{rest}}\Theta^\epsilon \equiv 0. \tag{3.7}$$

The function $\Psi^\epsilon_r(t, x)$ *satisfies*

$$|\hat{\Psi}^\epsilon_r(t, \boldsymbol{k})| \leqslant \frac{\boldsymbol{C}_0}{|\boldsymbol{k}|^4}, \qquad \mathbb{P}^\epsilon_{\text{large}}\Psi^\epsilon_r \equiv 0, \qquad \mathbb{P}^\epsilon_{\text{small}}\Psi^\epsilon_r \equiv 0. \tag{3.8}$$

If $\Psi(t,x) \in L^\epsilon$, then the norm $\|\Psi(t,x)\|_{L^\epsilon} = \min(\boldsymbol{C}_0)$, where $\boldsymbol{C}_0$ are such that for them all the conditions of definition 3 are satisfied.

Definition 4. $\mathbb{B}^\epsilon_{\boldsymbol{C}_0} = \{\Psi^\epsilon \in L^\epsilon \,|$ *for any* $t > 0$, $\|\Psi^\epsilon(t,x)\|_{L^\epsilon} \leqslant \boldsymbol{C}_0\}$ *is a ball of radius* $\boldsymbol{C}_0$.

Direct computations show that for any $\beta \geqslant 0$, L^ϵ can be embedded into Sobolev spaces $H^{1-\beta}$:

$$\|\Psi^\epsilon\|_{H^{1-\beta}} \leqslant C(1+\epsilon^\beta)\|\Psi^\epsilon\|_{L^\epsilon}. \tag{3.9}$$

Moreover, as $\epsilon \to 0$ for $\beta > 0$ the leading term of the $H^{1-\beta}$-norm of $\Psi^\epsilon \in L^\epsilon$ is determined by Ψ^ϵ_l, large-scale part of Ψ^ϵ:

$$\|\Psi^\epsilon - \Psi^\epsilon_l\|_{H^{1-\beta}} \leqslant \epsilon^\beta C\|\Psi^\epsilon\|_{L^\epsilon}. \tag{3.10}$$

We also have that $\|\Psi^\epsilon\|_{L^0} \leqslant \|\Psi^\epsilon\|_{L^\epsilon}$. Therefore, if the initial conditions satisfy $\|\Psi_0\|_{L^0} \leqslant \boldsymbol{C}_0$, then $\|\Psi_0\|_{L^\epsilon} \leqslant \boldsymbol{C}_0$.

For $\Theta^\epsilon(t,x)$, the form of the first term on the right-hand side of (3.7) arises, because the solution to

$$\left(\frac{1}{Re}\Delta + \mathbb{P}^\epsilon_{\text{small}}M^\epsilon\right)\Theta^\epsilon = -\mathbb{P}^\epsilon_{\text{small}}M^\epsilon\,\Xi^\epsilon$$

can be found from the geometric series expansion in small Re:

$$\Theta^\epsilon = Re\,\Theta^\epsilon_1 + Re^2\Theta^\epsilon_2 + \cdots, \qquad \Delta\Theta^\epsilon_1 = -M^\epsilon\,\Xi^\epsilon, \qquad \Delta\Theta^\epsilon_2 = -M^\epsilon\Theta^\epsilon_1, \ldots,$$

where (for sufficiently small Re) by (2.1) in lemma 1

$$|\hat{\Xi}^\epsilon(t,\boldsymbol{k})| \leqslant \frac{1}{2|\boldsymbol{k}||\boldsymbol{m}|^3}. \tag{3.11}$$

The form of the second term on the right-hand side of (3.7) arises due to the nonlinear estimates in appendix B.

3.3. Uniform boundedness of solutions

Consider a map $\mathbb{A}^\epsilon : \Psi^\epsilon \to \bar{\Psi}^\epsilon$, where $\bar{\Psi}^\epsilon$ is the solution of

$$\partial_t\bar{\Psi}^\epsilon(t,x) = \frac{1}{Re}\Delta\bar{\Psi}^\epsilon(t,x) + [M^\epsilon\Psi^\epsilon(t,x) - \mathbb{N}^\epsilon(\Psi^\epsilon(t,x))],$$
$$\bar{\Psi}^\epsilon(t=0,x) = \Psi_0(x).$$

The main advantage of lacunary spaces is that we can rewrite the map $\mathbb{A}^\epsilon$ and apply the fixed point argument for each of the functions Ψ^ϵ_l, Ξ^ϵ, Θ^ϵ, Ψ^ϵ_r separately in a simple manner as follows:

$$\begin{aligned}&\partial_t\bar{\Psi}^\epsilon_l(t,x) = \frac{1}{Re}\Delta\bar{\Psi}^\epsilon_l(t,x) + \mathbb{P}^\epsilon_{\text{large}}[M^\epsilon\,\Xi^\epsilon + M^\epsilon\Theta^\epsilon - \mathbb{N}^\epsilon(\Psi^\epsilon)],\\ &\bar{\Psi}^\epsilon_l(t=0,x) = \mathbb{P}^\epsilon_{\text{large}}\Psi_0(x),\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\partial_t\bar{\Xi}^\epsilon(t,x) = \frac{1}{Re}\Delta\bar{\Xi}^\epsilon(t,x) + M^\epsilon\Psi^\epsilon_l,\\ &\bar{\Xi}^\epsilon(t=0,x) = 0,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&\partial_t\bar{\Theta}^\epsilon(t,x) = \frac{1}{Re}\Delta\bar{\Theta}^\epsilon(t,x) + \mathbb{P}^\epsilon_{\text{small}}[M^\epsilon\,\Xi^\epsilon + M^\epsilon\Theta^\epsilon - \mathbb{N}^\epsilon(\Psi^\epsilon)],\\ &\bar{\Theta}^\epsilon(t=0,x) = \mathbb{P}^\epsilon_{\text{small}}\Psi_0(x),\end{aligned} \tag{3.14}$$

$$\begin{aligned}&\partial_t\bar{\Psi}^\epsilon_r(t,x) = \frac{1}{Re}\Delta\bar{\Psi}^\epsilon_r(t,x) + [M^\epsilon\Psi^\epsilon_r - \mathbb{P}^\epsilon_{\text{rest}}\mathbb{N}^\epsilon(\Psi^\epsilon)],\\ &\bar{\Psi}^\epsilon_r(t=0,x) = \mathbb{P}^\epsilon_{\text{rest}}\Psi_0(x).\end{aligned} \tag{3.15}$$

Lemma 4. *For any Re and ϵ, $\bar{\Psi}^\epsilon = \mathbb{A}^\epsilon(\Psi^\epsilon) \in L^\epsilon$, if $\Psi^\epsilon \in L^\epsilon$. Also, there exist sufficiently small Re_0 and ϵ_0 such that for any $Re < Re_0$ there exists $\boldsymbol{C}_0 = \boldsymbol{C}_0(Re)$, such that for any $\boldsymbol{C} \leqslant \boldsymbol{C}_0$, $\epsilon < \epsilon_0$, $\mathbb{A}^\epsilon : \mathbb{B}^\epsilon_{\boldsymbol{C}_0} \hookrightarrow \mathbb{B}^\epsilon_{\boldsymbol{C}_0}$, and*

$$\|\mathbb{A}^\epsilon(\Psi_1^\epsilon(t,x)) - \mathbb{A}^\epsilon(\Psi_2^\epsilon(t,x))\|_{L^\epsilon} \leqslant \tfrac{1}{2}\|\Psi_1^\epsilon(t,x) - \Psi_2^\epsilon(t,x)\|_{L^\epsilon}. \tag{3.16}$$

Therefore for such $Re < Re_0$ and $\epsilon < \epsilon_0$ $\Psi^\epsilon(t,x)$, the solution of the nonlinear modulation equation (1.4) with the initial conditions $\|\Psi_0(x)\|_{L^0} \leqslant \boldsymbol{C}_0$, is uniformly bounded: $\|\Psi^\epsilon(t,x)\|_{L^\epsilon} \leqslant \boldsymbol{C}_0$, for any $t \geqslant 0$.

Proof. Let $u(t)$ be the solution of an ordinary differential equation

$$\begin{aligned} &d_t u = -\frac{\alpha}{Re}u + f(t) + g(t), \qquad \alpha > 0, \quad Re > 0, \\ &u(0) = u_0, \qquad |f(t)| \leqslant \boldsymbol{c}_1\boldsymbol{C}_0, \quad |g(t)| \leqslant \boldsymbol{c}_2\boldsymbol{C}_0^2. \end{aligned} \tag{3.17}$$

Then

$$|u(t)| \leqslant |u_0| \exp\left(-\frac{\alpha t}{Re}\right) + \left(1 - \exp\left(-\frac{\alpha t}{Re}\right)\right)\frac{\boldsymbol{C}_0 Re}{\alpha}(\boldsymbol{c}_1 + \boldsymbol{c}_2\boldsymbol{C}_0). \tag{3.18}$$

Suppose $\boldsymbol{c}_1, \boldsymbol{c}_2$ are fixed constants, and Re, $\boldsymbol{C}_0$ are parameters. Regard $f(t)$ and $g(t)$ in (3.17) as linear and nonlinear (quadratic) in $\boldsymbol{C}_0$ forcing terms. The goal is to find *first* the critical value Re_0 of the parameter Re so that, for any $Re < Re_0$, we *then* can find $\boldsymbol{C}_0 = \boldsymbol{C}_0(Re)$ such that if $|u_0| \leqslant \boldsymbol{C}_0/\alpha$, then $|u(t)| \leqslant \boldsymbol{C}_0/\alpha$ for all $t > 0$. From (3.18), this can be done by first disregarding the nonlinear forcing term and setting $Re_0 = 1/\boldsymbol{c}_1$ and then setting $\boldsymbol{C}_0 = (1 - Re\,\boldsymbol{c}_1)/(Re\,\boldsymbol{c}_2)$. The proof of uniform boundedness is an application of this observation to every Fourier coefficient in (3.15), (3.12) and (3.14). There, in order to determine Re_0 we disregard the nonlinear forcing terms, then the linear forcing comes from the operator of eddy viscosity M^ϵ only. Then $\hat{\bar{\Psi}}^\epsilon_l(t,\boldsymbol{m}) \leqslant \boldsymbol{C}_0/|\boldsymbol{m}|^4$ for sufficiently small Re_0, by the estimates from the linearized case section 2.3. $\bar{\Xi}^\epsilon(t,x)$ satisfies its condition in the definition of the lacunary spaces by construction. Therefore we only need to check the estimates for $\bar{\Psi}^\epsilon_r(t,x)$ and $\bar{\Theta}^\epsilon(t,x)$. We recall that, due to the form of the operator of eddy viscosity, each Fourier coefficient $\hat{\bar{\Psi}}^\epsilon(t,\bar{\boldsymbol{k}})$, $\bar{\boldsymbol{k}} = (\boldsymbol{m},\bar{\boldsymbol{n}})_\epsilon$ is determined by only four Fourier coefficients $\hat{\Psi}^\epsilon(t,\boldsymbol{k})$, $\boldsymbol{k} = (\boldsymbol{m},\boldsymbol{n})_\epsilon$, where $\boldsymbol{n}$ are the nearest neighbours of $\bar{\boldsymbol{n}}$: $\boldsymbol{n} = \bar{\boldsymbol{n}} + (\pm 1, \pm 1)$. In most of the cases it implies that if $Re \leqslant Re_0$, then

$$|\hat{\bar{\Phi}}^\epsilon(t,\bar{\boldsymbol{k}})| \leqslant \frac{Re_0}{|\bar{\boldsymbol{k}}|^2}\sum_{\boldsymbol{k}=\bar{\boldsymbol{k}}+(\pm 1,\pm 1)/\epsilon} \boldsymbol{C}_{\boldsymbol{k}} \max_{0\leqslant\tau\leqslant t}|\hat{\Phi}^\epsilon(\tau,\boldsymbol{k})|, \tag{3.19}$$

where $\boldsymbol{C}_{\boldsymbol{k}}$ are some constants and $\Phi^\epsilon(t,x)$ is either $\Psi^\epsilon_r(t,x)$ or $\Theta^\epsilon(t,x)$. The only exceptions are when $\bar{\boldsymbol{n}} = (\pm 1, \pm 1) + (\pm 1, \pm 1)$, $\|\!|\bar{\boldsymbol{n}}|\!\| > 1$, because then $\hat{\bar{\Theta}}^\epsilon(t,x)$ is also forced by $\bar{\Xi}^\epsilon(t,x)$. Consider first the generic case. Then in (3.19) $\boldsymbol{C}_{\boldsymbol{k}} \leqslant ||\boldsymbol{k}|^2 - 2/\epsilon^2|/(|\bar{\boldsymbol{k}}|\epsilon)$. Since $C_1|\boldsymbol{k}| \leqslant |\bar{\boldsymbol{k}}| \leqslant C_2|\boldsymbol{k}|$ (where $C_1 = C_1(\sigma)$, $C_2 = C_2(\sigma)$ for $\Psi^\epsilon_r(t,x)$), we have that $\boldsymbol{C}_{\boldsymbol{k}} \leqslant |\boldsymbol{k}|^2$ for $\Theta^\epsilon(t,x)$, and $\boldsymbol{C}_{\boldsymbol{k}} \leqslant C(\sigma)|\boldsymbol{k}|^2$ for $\Psi^\epsilon_r(t,x)$. Therefore we can choose a sufficiently small Re_0, so that the estimates for $\bar{\Psi}^\epsilon_r(t,x)$ and $\bar{\Theta}^\epsilon(t,x)$ hold. In the exceptional cases the argument is similar, when we observe that for $\bar{\boldsymbol{n}} = (\pm 1, \pm 1) + (\pm 1, \pm 1)$, $\|\!|\bar{\boldsymbol{n}}|\!\| > 1$, in the sum on the right-hand side of (3.19) appears an extra term: $\boldsymbol{C}_{\boldsymbol{k}} \max_{0\leqslant\tau\leqslant t}|\hat{\bar{\Xi}}^\epsilon(\tau,\boldsymbol{k})|$ with a better estimate on the constant: $\boldsymbol{C}_{\boldsymbol{k}} \leqslant |\boldsymbol{m}||\boldsymbol{k}|$. Thus we have shown that there exists a critical Re_0. Observe that, by lemma 18, the nonlinear corrections do not affect $\Xi^\epsilon(t,x)$, and therefore the symmetry condition (1.20) for the fully nonlinear map $\mathbb{A}^\epsilon$ is satisfied, if it is satisfied for its linear part. Therefore the existence of $\boldsymbol{C}_0(Re)$ follows from lemma 18 in appendix B. Estimate (3.16) is proved similarly. ∎

3.4. Homogenization

The proof of the homogenization of the nonlinear modulation equation relies on repeated use of the idea of the reduction to a simpler model. More specifically, we replace one differential equation by another simpler equation which is asymptotically in ϵ equivalent to the first one, and then view the error as a small forcing. For clarity, we subdivided the presentation into four parts. It is fairly simple to show by standard estimates that the solution of an equation with small forcing is close to the solution of the same equation without forcing. The only notable and unexpected exception is the reduction we describe in part 1. The error of this reduction is analysed in part 2, where we use the idea that supposes there are two contraction maps $\mathbb{A}$ and $\mathbb{A}_0$ acting on the same Banach space, such that inequality $\|u - u_0\| < \boldsymbol{c}_0$ implies $\|\mathbb{A}u - \mathbb{A}_0 u_0\| < \boldsymbol{c}_0$, then the fixed points of these maps will also satisfy the same estimate $\|\tilde{u} - \tilde{u}_0\| < \boldsymbol{c}_0$. In part 3 we estimate the error that may arise from the short-time initial growth of the solution. Note that, since the linearized problem can be reduced to a problem where the large-scale part of the solution has only one nonzero Fourier coefficient, the need for this estimate arises only in the nonlinear case. Part 4 is basically matching asymptotics and it is similar to the proof of homogenization in the linearized case.

3.4.1. Reduction to a simpler problem. Observe that for a function $\Psi^\epsilon(t, x) \in L^\epsilon$ there are (see appendix B) 16 terms with different estimates on their Fourier coefficients, that arise if we expand $\mathbb{N}^\epsilon(\Psi^\epsilon(t, x))$ using the lacunary expansion. In order to prove the homogenization, we need *a priori* estimates on their *time derivatives*. While most of these terms are negligible as $\epsilon \to 0$, *a priori* bounds on their time derivatives are such that they do not guarantee that $\partial_t \Psi^\epsilon(t, x) \in L_2$. A way to simplify the analysis is to introduce an auxiliary function $\Psi_0^\epsilon(t, x)$, such that it satisfies the (nonlinear partial differential) equation that has only the 'significant' terms of the nonlinear modulation equation (1.4). More specifically, define

$$\Psi_0^\epsilon(t, x) = \Psi_{l,0}^\epsilon(t, x) + \Xi_0^\epsilon(t, x) + \Theta_0^\epsilon(t, x), \qquad \mathbb{P}_{\text{rest}}^\epsilon \Psi_0^\epsilon(t, x) \equiv 0 \tag{3.20}$$

as the solution of

$$\begin{aligned} &\partial_t \Psi_{l,0}^\epsilon + \mathbb{P}_{\text{large}}^\epsilon[\mathbb{N}^\epsilon(\Psi_{l,0}^\epsilon) + \mathbb{N}^\epsilon(\Xi_0^\epsilon)] = \frac{1}{Re}\Delta \Psi_{l,0}^\epsilon + M^\epsilon \Psi_{l,0}^\epsilon, \\ &\Psi_{l,0}^\epsilon(t = 0, x) = \mathbb{P}_{\text{large}}^\epsilon \Psi_0(x), \end{aligned} \tag{3.21}$$

$$\begin{aligned} &\partial_t \Xi_0^\epsilon = \frac{1}{Re}\Delta \Xi_0^\epsilon + M^\epsilon \Psi_{l,0}^\epsilon, \\ &\Xi_0^\epsilon(t = 0, x) = 0, \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\partial_t \Theta_0^\epsilon + \mathbb{P}_{\text{small}}^\epsilon \mathbb{N}^\epsilon(\Psi_{l,0}^\epsilon, \Xi_0^\epsilon) = \frac{1}{Re}\Delta \Theta_0^\epsilon + \mathbb{P}_{\text{small}}^\epsilon M^\epsilon[\Xi_0^\epsilon + \Theta_0^\epsilon], \\ &\Theta_0^\epsilon(t = 0, x) = 0. \end{aligned} \tag{3.23}$$

Following the steps in sections 3.2 and 3.3 and using lemma 19 in appendix B, we have that $\Psi_0^\epsilon(t, x)$ is a fixed point solution of a (new) map $\mathbb{A}_0^\epsilon : L_0^\epsilon \to L_0^\epsilon$ on $\mathbb{B}_{\boldsymbol{C}_0}^\epsilon \subset L_0^\epsilon$, a ball in a new lacunary space L_0^ϵ (without loss of generality $\boldsymbol{C}_0$ can be chosen so that $\Psi^\epsilon(t, x)$ is a fixed point solution of $\mathbb{A}^\epsilon$ on $\mathbb{B}_{\boldsymbol{C}_0}^\epsilon \subset L^\epsilon$). The new lacunary space L_0^ϵ contains functions with improved bounds on the Fourier coefficients of Θ_0^ϵ:

$$|\hat{\Theta}_0^\epsilon(t, \boldsymbol{k})| \leqslant \frac{\boldsymbol{C}_0}{2^{\|\boldsymbol{n}\|}|\boldsymbol{m}|^2|\boldsymbol{k}|^2}, \tag{3.24}$$

$\mathbb{P}_{\text{rest}}^\epsilon \Psi_0^\epsilon(t, x) \equiv 0$, the bounds on $\Psi_{l,0}^\epsilon$ and Ξ_0^ϵ are the same. The new Banach space L_0^ϵ is naturally embedded in L^ϵ, and therefore we also can view the map $\mathbb{A}_0^\epsilon : L^\epsilon \to L^\epsilon$.

3.4.2. Error of the reduction. The next step is to show that $\Psi_0^\epsilon(t,x)$ is a good approximation to $\Psi^\epsilon(t,x)$, that is $\|\Psi_0^\epsilon(t,x)-\Psi^\epsilon(t,x)\|_{H^{1-\beta}}\to 0$, as $\epsilon\to 0$. Indeed, any $\Psi^\epsilon(t,x)$, $\|\Psi^\epsilon(t,x)\|_{L^\epsilon}\leqslant \boldsymbol{C}_0$ can be decomposed into

$$\Psi^\epsilon(t,x)=\Phi^\epsilon(t,x)+\Upsilon^\epsilon(t,x),\qquad \|\Phi^\epsilon(t,x)\|_{L_0^\epsilon}\leqslant \boldsymbol{C}_0,$$

$$|\hat{\Upsilon}^\epsilon(t,\boldsymbol{k})|\leqslant\begin{cases}\dfrac{\boldsymbol{C}_0}{|\boldsymbol{k}|^3|\boldsymbol{m}|}, & |\boldsymbol{m}|\leqslant\frac{\sigma}{\epsilon},\quad \boldsymbol{n}\neq(0,0),\\ 0, & |\boldsymbol{k}|\leqslant\dfrac{\sigma}{\epsilon},\\ \dfrac{\boldsymbol{C}_0}{|\boldsymbol{k}|^4}, & \text{otherwise}.\end{cases}\tag{3.25}$$

Computing $\mathbb{A}_0^\epsilon\Psi_0^\epsilon-\mathbb{A}^\epsilon\Psi^\epsilon$ by following the steps in section 3.3 we have that there is a sufficiently small $\boldsymbol{C}_0$ (again, without loss of generality, choose $\boldsymbol{C}_0$ so that $\mathbb{A}_0^\epsilon$ and $\mathbb{A}^\epsilon$ are contractions on $\mathbb{B}_{\boldsymbol{C}_0}^\epsilon\subset L^\epsilon$ and $\mathbb{B}_{\boldsymbol{C}_0}^\epsilon\subset L_0^\epsilon$, respectively), so that if $\Psi^\epsilon=\Phi^\epsilon+\Upsilon^\epsilon$ as in (3.25) and

$$\|\Phi^\epsilon(t,x)-\Psi_0^\epsilon(t,x)\|_{L_0^\epsilon}\leqslant\epsilon\boldsymbol{C}_0,\tag{3.26}$$

then for

$$\mathbb{A}_0^\epsilon\Psi_0^\epsilon(t,x)=\bar{\Psi}_0^\epsilon(t,x),\qquad \mathbb{A}^\epsilon\Psi^\epsilon(t,x)=\bar{\Psi}^\epsilon(t,x),$$

we have that $\bar{\Psi}^\epsilon(t,x)$ also can be represented as in (3.25) and

$$\|\bar{\Phi}^\epsilon(t,x)-\bar{\Psi}_0^\epsilon(t,x)\|_{L_0^\epsilon}\leqslant\epsilon\boldsymbol{C}_0.$$

It implies that the fixed points of the maps $\mathbb{A}_0^\epsilon$, $\mathbb{A}^\epsilon$ also satisfy (3.25) with (3.26). Then, by (3.9) and (3.26), $\Psi^\epsilon(t,x)$, the solution of the nonlinear modulation equation (1.4), and $\Psi_0^\epsilon(t,x)$, the solution of (3.21)–(3.23), satisfy

$$\|\Psi_0^\epsilon-\Psi^\epsilon\|_{H^{1-\beta}}\leqslant C(1+\epsilon^\beta)\|\Psi_0^\epsilon-\Phi^\epsilon\|_{L_0^\epsilon}+\|\Upsilon^\epsilon\|_{H^1}\leqslant\epsilon C\boldsymbol{C}_0\to 0,\qquad \text{as }\epsilon\to 0.$$

3.4.3. Relation to the effective equations. By (3.10) the $H^{1-\beta}$-norms of $\Xi_0^\epsilon(t,x)$ and $\Theta_0^\epsilon(t,x)$ are negligible if $\beta>0$:

$$\|\Psi_0(t,x)-\Psi_{l,0}^\epsilon(t,x)\|_{H^{1-\beta}}\to 0,$$

as $\epsilon\to 0$. Therefore, similar to the proof of theorem 1 in section 2.4, the next step is to prove that $\Psi_{l,0}^\epsilon(t,x)$, the large-scale part of $\Psi_0^\epsilon(t,x)$, satisfies the effective nonlinear equation (1.7) with some additional small forcing:

$$\begin{aligned}&\partial_t\Psi_{l,0}^\epsilon(t,x)+\mathbb{P}_{\text{large}}^\epsilon\mathbb{N}(\Psi_{l,0}^\epsilon(t,x))=\frac{1}{Re}\Delta\Psi_{l,0}^\epsilon(t,x)+M\Psi_{l,0}^\epsilon(t,x)+F^\epsilon(t,x),\\ &\Psi_{l,0}^\epsilon(t=0,x)=\mathbb{P}_{\text{large}}^\epsilon\Psi_0(x),\end{aligned}\tag{3.27}$$

where

$$|\hat{F}^\epsilon(t,\boldsymbol{m})|\leqslant\frac{C}{|\boldsymbol{m}|^2},\qquad \mathbb{P}_{\text{small}}^\epsilon F^\epsilon(t,x)\equiv 0,\qquad \mathbb{P}_{\text{rest}}^\epsilon F^\epsilon(t,x)\equiv 0\tag{3.28}$$

and there exists $t_0^\epsilon\to 0$, as $\epsilon\to 0$, such that for any t, $t\geqslant t_0^\epsilon$

$$|\hat{F}^\epsilon(t,\boldsymbol{m})|\leqslant\epsilon C.\tag{3.29}$$

Here we use the bootstrapping argument. Step by step we convert (3.21)–(3.23) into (3.27) with (3.28) and (3.29). Here are the steps:

(i) $|\partial_t\hat{\Psi}_{l,0}^\epsilon(t,\boldsymbol{m})|\leqslant C/|\boldsymbol{m}|^2$,

(ii) there exists $t_0^\epsilon\to 0$, as $\epsilon\to 0$, such that for $t\geqslant t_0$

$$\Xi_0^\epsilon(t,x)=\Psi^1(t,x)+\zeta_1^\epsilon(t,x),\tag{3.30}$$

$$\Psi^1(t,x)=-Re\Delta^{-1}M^\epsilon\Psi_l^\epsilon,\qquad |\hat{\zeta}_1^\epsilon(t,\boldsymbol{k})|\leqslant\frac{C}{|\boldsymbol{k}|^3|\boldsymbol{m}|},\qquad |\partial_t\Xi_0^\epsilon(t,\boldsymbol{k})|\leqslant\frac{C}{|\boldsymbol{k}||\boldsymbol{m}|},$$

(iii) $|\partial_t \hat{\Theta}_0^\epsilon(t, \boldsymbol{k})| \leqslant C/(2^{\|\boldsymbol{n}\|}|\boldsymbol{k}|^2)$,

(iv) there exists $t_0^\epsilon \to 0$, as $\epsilon \to 0$, such that for $t \geqslant t_0$

$$\Theta_0^\epsilon(t,x) = \Psi^2(t,x) + \zeta_3^\epsilon(t,x),$$

$$\Psi^2(t,x) = -\left(\frac{1}{Re}\Delta + M^\epsilon\right)^{-1} \mathbb{P}^\epsilon_{\mathrm{small}}[M^\epsilon \Psi^1 - \mathbb{N}^\epsilon(\Psi^\epsilon_{l,0}, \Psi^1)], \tag{3.31}$$

$$\zeta_3^\epsilon(t,x) = -\left(\frac{1}{Re}\Delta + M^\epsilon\right)^{-1} \mathbb{P}^\epsilon_{\mathrm{small}}[M^\epsilon \zeta_1^\epsilon - \mathbb{N}^\epsilon(\Psi^\epsilon_{l,0}, \zeta_1^\epsilon)] + \zeta_2^\epsilon, \qquad |\hat{\zeta}_2^\epsilon(t,\boldsymbol{k})| \leqslant \frac{C}{|\boldsymbol{k}|^4},$$

(v) $|\hat{\zeta}_3^\epsilon(t,\boldsymbol{k})| \leqslant C/(|\boldsymbol{m}||\boldsymbol{k}|^3)$,

(vi) there exists $t_0^\epsilon \to 0$, as $\epsilon \to 0$, such that for $t \geqslant t_0$

$$\begin{aligned}
&\partial_t \Psi^\epsilon_{l,0}(t,x) + \mathbb{P}^\epsilon_{\mathrm{large}}[\mathbb{N}^\epsilon(\Psi^\epsilon_{l,0}(t,x)) + \mathbb{N}^\epsilon(\Psi^1(t,x))] \\
&\qquad = \frac{1}{Re}\Delta\Psi^\epsilon_{l,0}(t,x) + \mathbb{P}^\epsilon_{\mathrm{large}} M^\epsilon[\Psi^1(t,x) + \Psi^2(t,x)] + F^\epsilon(t,x), \\
&\Psi^\epsilon_{l,0}(t=0,x) = \mathbb{P}^\epsilon_{\mathrm{large}}\Psi_0(x), \\
&F_1^\epsilon(t,x) = \mathbb{P}^\epsilon_{\mathrm{large}}(M^\epsilon[\zeta_1^\epsilon + \zeta_3^\epsilon] - \mathbb{N}^\epsilon(\Psi^1, \zeta_3^\epsilon) - \mathbb{N}^\epsilon(\zeta_3^\epsilon, \Psi^1) - \mathbb{N}^\epsilon(\zeta_3^\epsilon)), \\
&|\hat{F}_1^\epsilon(t,\boldsymbol{m})| \leqslant \frac{\epsilon C}{|\boldsymbol{m}|}.
\end{aligned}$$

(vii) $\Psi^\epsilon_{l,0}(t,x)$ satisfies (3.27) with (3.28) and (3.29).

Using the nonlinear estimates in appendix B, and linear estimates in sections 2.1–2.3, we have (i) by direct computations. (i) implies (ii) by lemma 1. Differentiate (3.23) with respect to t. Then $\Theta^\epsilon_{dt}(t,x) = \partial_t\Theta^\epsilon_0(t,x)$ satisfies

$$\begin{aligned}
&\partial_t\Theta^\epsilon_{dt} = \frac{1}{Re}\Delta\Theta^\epsilon_{dt} + \mathbb{P}^\epsilon_{\mathrm{small}} M^\epsilon\Theta^\epsilon_{dt} + \Phi_2^\epsilon(t,x) + \Phi_1^\epsilon(t,x), \\
&\Phi_1^\epsilon = -\mathbb{P}^\epsilon_{\mathrm{small}}[\mathbb{N}^\epsilon(\partial_t\Psi^\epsilon_{l,0}, \Xi_0^\epsilon) + \mathbb{N}^\epsilon(\Psi^\epsilon_{l,0}, \partial_t\Xi_0^\epsilon)], \qquad \Phi_2^\epsilon = \mathbb{P}^\epsilon_{\mathrm{small}} M^\epsilon\partial_t\Xi_0^\epsilon, \\
&\Theta^\epsilon_{dt}(t=0,x) = 0.
\end{aligned} \tag{3.32}$$

The only nonzero Fourier coefficients $\hat{\Phi}_1^\epsilon(t,\boldsymbol{k})$, $\boldsymbol{k} = (\boldsymbol{m},\boldsymbol{n})_\epsilon$, are when $\boldsymbol{n} = (\pm1,\pm1)$ and by lemma 10 in appendix B,

$$|\hat{\Phi}_1^\epsilon(t,\boldsymbol{k})| \leqslant C. \tag{3.33}$$

The only nonzero Fourier coefficients $\hat{\Phi}_2^\epsilon(t,\boldsymbol{k})$, $\boldsymbol{k} = (\boldsymbol{m},\boldsymbol{n})_\epsilon$, are when $\boldsymbol{n} \neq (0,0)$ and it is a nearest neighbour of one of $\bar{\boldsymbol{n}} = (\pm1,\pm1)$, that is when $\boldsymbol{n} = (\pm1,\pm1) + (\pm1,\pm1)$, and there, by the estimate on $\partial_t\Xi_0^\epsilon(t,x)$ in (ii), we have

$$|\hat{\Phi}_2^\epsilon(t,\boldsymbol{k})| \leqslant C. \tag{3.34}$$

From (3.33) and (3.34), reformulating (3.32) as a fixed point solution of a linear map and following the argument in the proof of lemma 4 in section 3.3, we have (iii). (iii) implies (iv) by estimation of the terms in (3.23). Again, following the argument in the proof of lemma 4, and using the estimates in lemma 17 of appendix B, we have (v) and the estimate for $F_1^\epsilon(t,x)$ in (vi). Observe that $\Psi^1(t,x)$ (in (3.30)) and $\Psi^2(t,x)$ (in (3.4.3)) are exactly the functions, that come from the multiscale analysis of [35] (see appendix A). Therefore, (vi) implies (vii) by lemmas 5–7 (condition (A.8) is satisfied by lemmas 1 and 2; conditions (A.9) and (A.10) are satisfied by definition of L_0^ϵ).

3.4.4. Error of the effective equation with forcing. Finally, since for $\Psi(t,x)$, the solution of the effective nonlinear equation (1.7) in L^0,

$$\|\mathbb{P}^{\epsilon}_{\mathrm{large}}\Psi - \Psi\|_{H^1} \leqslant \boldsymbol{C}_0 \left(\sum_{\boldsymbol{m}\geqslant\sigma/\epsilon} \frac{1}{|\boldsymbol{m}|^6}\right)^{1/2} \leqslant \epsilon^2 \boldsymbol{C}_0,$$

we only need to show that, if (3.28) and (3.29) holds, then $\|\Phi^{\epsilon}\|_{H^{1-\beta}} \to 0$, as $\epsilon \to 0$, where $\Phi^{\epsilon}(t,x) = \mathbb{P}^{\epsilon}_{\mathrm{large}}\Psi - \Psi^{\epsilon}_{l,0}$.

Suppose $|\hat{\Phi}^{\epsilon}(t,\boldsymbol{m})| \leqslant \boldsymbol{C}_1(\epsilon)/|\boldsymbol{m}|^2$ with $\boldsymbol{C}_1(\epsilon) \to 0$ as $\epsilon \to 0$. Then

$$\|\Phi^{\epsilon}(t,x)\|_{H^{1-\beta}} \leqslant \boldsymbol{C}_1(\epsilon) \left(\sum_{\boldsymbol{m}} \frac{1}{|\boldsymbol{m}|^{2+2\beta}}\right)^{1/2} \leqslant C(\beta)\boldsymbol{C}_1(\epsilon) \to 0, \qquad \text{as } \epsilon \to 0.$$

Therefore we complete the proof of theorem 2 by showing that

$$|\hat{\Phi}^{\epsilon}(t,\boldsymbol{m})| \leqslant \frac{C}{|\boldsymbol{m}|^2}(t_0^{\epsilon} + \epsilon). \tag{3.35}$$

If $0 \leqslant t \leqslant t_0^{\epsilon}$, then by (3.28) $|d_t\hat{\Phi}^{\epsilon}(t,\boldsymbol{m})| \leqslant C/|\boldsymbol{m}|^2$, and therefore (3.35) holds. Suppose $t \geqslant t_0^{\epsilon}$. Since

$$\begin{aligned}
&\mathbb{N}(\Psi(t,x)) - \mathbb{N}(\Psi^{\epsilon}_{l,0}(t,x)) = G^{\epsilon}(\Phi^{\epsilon}(t,x)) + F_2^{\epsilon}(t,x),\\
&G^{\epsilon}(\Phi^{\epsilon}(t,x)) = \mathbb{P}^{\epsilon}_{\mathrm{large}}[\mathbb{N}(\mathbb{P}^{\epsilon}_{\mathrm{large}}\Psi(t,x), \Phi^{\epsilon}(t,x)) + \mathbb{N}(\Phi^{\epsilon}(t,x), \Psi^{\epsilon}_{l,0}(t,x))],\\
&F_2^{\epsilon}(t,x) = \mathbb{P}^{\epsilon}_{\mathrm{large}}[\mathbb{N}(\Psi(t,x)) - \mathbb{N}(\mathbb{P}^{\epsilon}_{\mathrm{large}}\Psi(t,x))],
\end{aligned}$$

for $t \geqslant t_0^{\epsilon}$, $\Phi^{\epsilon}(t,x)$ is the solution of a linear equation

$$\begin{aligned}
&\partial_t\Phi^{\epsilon}(t,x) = \frac{1}{Re}\Delta\Phi^{\epsilon}(t,x) + M\Phi^{\epsilon}(t,x) - G^{\epsilon}(\Phi^{\epsilon}(t,x)) + F_3^{\epsilon}(t,x),\\
&F_3^{\epsilon}(t,x) = F^{\epsilon}(t,x) - F_2^{\epsilon}(t,x),\\
&|\hat{\Phi}^{\epsilon}(t=t_0^{\epsilon},\boldsymbol{m})| \leqslant \frac{C}{|\boldsymbol{m}|^2}t_0^{\epsilon}.
\end{aligned} \tag{3.36}$$

By lemma 8, $|\hat{F}_2^{\epsilon}(t,\boldsymbol{m})| \leqslant \epsilon C$, therefore $|\hat{F}_3^{\epsilon}(t,\boldsymbol{m})| \leqslant \epsilon C$. Denote

$$\boldsymbol{C}^{\epsilon}_{\Phi}(t) = \max_{\tau\in[0,t]}\max_{\boldsymbol{m}}(|\boldsymbol{m}|^2|\hat{\Phi}^{\epsilon}_l(\tau,\boldsymbol{m})|). \tag{3.37}$$

By lemma 9, $|\widehat{G^{\epsilon}\Phi^{\epsilon}}(t,\boldsymbol{m}))| \leqslant C\boldsymbol{C}_0\boldsymbol{C}^{\epsilon}_{\Phi}(t)$. Therefore if $Re < Re_0$ there is a small constant $\boldsymbol{C}_0(Re)$ (again, this $\boldsymbol{C}_0$ may be smaller than $\boldsymbol{C}_0$ above, and we take the smallest of the two), such that each Fourier coefficient of $\Phi^{\epsilon}(t,x)$ satisfies a differential inequality

$$\begin{aligned}
&d_t|\hat{\Phi}^{\epsilon}(t,\boldsymbol{m})| \leqslant -\frac{1}{Re}|\boldsymbol{m}|^2|\hat{\Phi}^{\epsilon}(t,\boldsymbol{m})| + \frac{1}{2Re}\boldsymbol{C}^{\epsilon}_{\Phi}(t) + \epsilon C,\\
&|\hat{\Phi}^{\epsilon}(t=t_0^{\epsilon},\boldsymbol{m})| \leqslant \frac{C}{|\boldsymbol{m}|^2}t_0^{\epsilon}.
\end{aligned} \tag{3.38}$$

Integrating (3.38) we have that for any $\boldsymbol{m}$

$$|\hat{\Phi}^{\epsilon}(t,\boldsymbol{m})| \leqslant \frac{C}{|\boldsymbol{m}|^2}t_0^{\epsilon} + \frac{1}{2}\frac{\boldsymbol{C}^{\epsilon}_{\Phi}(t)}{|\boldsymbol{m}|^2} + \epsilon C\frac{Re}{|\boldsymbol{m}|^2}$$

and therefore

$$\frac{\boldsymbol{C}^{\epsilon}_{\Phi}(t)}{|\boldsymbol{m}|^2} \leqslant \frac{C}{|\boldsymbol{m}|^2}t_0^{\epsilon} + \frac{1}{2}\frac{\boldsymbol{C}^{\epsilon}_{\Phi}(t)}{|\boldsymbol{m}|^2} + \epsilon C\frac{Re}{|\boldsymbol{m}|^2} \leqslant (t_0^{\epsilon} + \epsilon)\frac{C}{|\boldsymbol{m}|^2},$$

which, using (3.37), gives (3.35). This completes the proof of theorem 2.

Acknowledgments

We are grateful to G Papanicolaou, L Berlyand and J Mattingly for reading a draft of this paper and making remarks about it.

Appendix A. Summary of the multiscale asymptotics

Here we give a brief summary of the (formal) derivation of the effective equations (1.7) and (1.10) from (1.4) and (1.6), respectively. Our derivation is different from [35] in two respects. Namely, $\Psi_l^\epsilon(t,x)$ is not a solution of the effective equations (as it was in [35]) and we do not replace the Laplacian by its two-scale asymptotic expansion. These modifications were made for consistency with our Ψ_l^ϵ, Ξ^ϵ, Θ^ϵ expansion.

Suppose $\Psi^\epsilon = \Psi_l^\epsilon + \Psi^1 + \Psi^2 + \mathrm{O}(\epsilon^3)$, where $\Psi^k = \mathrm{O}(\epsilon^k)$, $k = 1, 2$ as $\epsilon \to 0$. In the linear and the nonlinear cases $\Psi^1(t,x)$, the first-order corrector, satisfies

$$\frac{1}{Re}\Delta\Psi^1 = -M^\epsilon\Psi_l^\epsilon. \tag{A.1}$$

The second-order corrector $\Psi^2(t,x)$ in the nonlinear case satisfies

$$\left(\frac{1}{Re}\Delta + M^\epsilon\right)\Psi^2 = -\mathbb{P}^\epsilon_{\mathrm{small}}[M^\epsilon\Psi^1 - \mathbb{N}^\epsilon(\Psi_l^\epsilon, \Psi^1)]. \tag{A.2}$$

In the linear case the last term in (A.2) should be dropped, and Ψ^2 satisfies

$$\left(\frac{1}{Re}\Delta + M^\epsilon\right)\Psi^2 = -\mathbb{P}^\epsilon_{\mathrm{small}}M^\epsilon\Psi^1. \tag{A.3}$$

Matching asymptotic expansions for $\Psi_l^\epsilon(t,x)$ we have that

$$\begin{aligned}&\partial_t\Psi_l^\epsilon = \frac{1}{Re}\Delta\Psi_l^\epsilon + \mathbb{P}^\epsilon_{\mathrm{large}}[M^\epsilon\Psi^1 + M^\epsilon\Psi^2 - \mathbb{N}^\epsilon(\Psi_l^\epsilon) - \mathbb{N}^\epsilon(\Psi^1)] + \mathrm{O}(\epsilon),\\ &\Psi_l^\epsilon(t=0,x) = \mathbb{P}^\epsilon_{\mathrm{large}}\Psi_0(x).\end{aligned} \tag{A.4}$$

(A.1) can be solved explicitly in Fourier series, and $\mathbb{P}^\epsilon_{\mathrm{large}}M^\epsilon\Psi^1$ in (A.4) gives rise to the term

$$-\frac{Re}{8}((\nabla_1 + \delta\nabla_2)^2 + (\delta\nabla_1 + \nabla_2)^2)\Psi + \frac{Re}{2}(1+\delta^2)\Delta^{-1}(\nabla_2^2 - \nabla_1^2)^2\Psi$$

in (1.9). Since (A.2) is a linear equation, the solution of it is a sum of two solutions: $\Psi^2 = \Psi_1^2$, the solution of (A.3), and $\Psi^2 = \Psi_2^2$, the solution of

$$\left(\frac{1}{Re}\Delta + M^\epsilon\right)\Psi^2 = \mathbb{P}^\epsilon_{\mathrm{small}}\mathbb{N}^\epsilon(\Psi_l^\epsilon, \Psi^1). \tag{A.5}$$

Due to cancellations, Ψ_2^2 can also be computed analytically explicitly. Then $\mathbb{P}^\epsilon_{\mathrm{large}}M^\epsilon\Psi_2^2 = -2\mathbb{P}^\epsilon_{\mathrm{large}}\mathbb{N}^\epsilon(\Psi^1)$, and $\mathbb{P}^\epsilon_{\mathrm{large}}M^\epsilon\Psi_2^2 + \mathbb{P}^\epsilon_{\mathrm{large}}\mathbb{N}^\epsilon(\Psi^1)$ gives rise to the term

$$\frac{Re^2}{8}\Delta^{-1}(\nabla_2^2 - \nabla_1^2)[\delta((\nabla_1\Psi)^2 + (\nabla_2\Psi)^2) + (1+\delta^2)\nabla_1\Psi\nabla_2\Psi]$$

in (1.8).

The term Ψ_1^2 cannot be found explicitly analytically. In particular, the first term of the asymptotic expansion in ϵ of Ψ_1^2, must be determined by $\psi(y)$, the solution of the following elliptic periodic boundary value problem in fast variables $y = (y_1, y_2)$

$$\begin{aligned}&\frac{1}{Re}\Delta\Delta\psi(y) - J_{yy}\left(\phi(y), (\Delta+2)\psi(y) - \frac{Re}{2}\phi\right) = 0,\\ &\phi(y) = \cos y_1\cos y_2 + \delta\sin y_1\sin y_2.\end{aligned} \tag{A.6}$$

Using the solution of (A.6), $\mathbb{P}^{\epsilon}_{\text{large}} M^{\epsilon} \Psi_1^2$ gives rise to the term

$$\nu' \Delta^{-1} (\nabla_2^2 - \nabla_1^2)^2 \Psi$$

in (1.9), where

$$\nu' = -4\langle \phi(y)\psi(y)\rangle = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \phi(y)\psi(y)\, \mathrm{d}y_1\, \mathrm{d}y_2. \tag{A.7}$$

The $O(\epsilon^3)$ terms do not affect the effective equation and can be neglected.

From the multiscale analysis we have the error estimates which are needed for the bootstrapping arguments. These estimates are summarized in the following three lemmas.

Lemma 5. *Suppose* $\Psi_l^{\epsilon}(t,x) = a^{\epsilon}(t)\exp(\mathrm{i}\boldsymbol{m}\cdot x)$*,* Ψ^1 *satisfies (A.1),* Ψ^2 *satisfies (A.3), then*

$$M\Psi_l^{\epsilon}(t,x) - \mathbb{P}^{\epsilon}_{\text{large}} M^{\epsilon}[\Psi^1 + \Psi^2] = \hat{f}^{\epsilon}(\boldsymbol{m},(t))\exp(\mathrm{i}\boldsymbol{m}\cdot x).$$

If

$$\|\Psi^1\|_{L^2} \leqslant \boldsymbol{C}_1 \epsilon |\boldsymbol{m}||a^{\epsilon}(t)|, \qquad \|\Psi^2\|_{L^2} \leqslant \boldsymbol{C}_2 \epsilon^2 |\boldsymbol{m}|^2 |a^{\epsilon}(t)|, \tag{A.8}$$

then

$$|\hat{f}^{\epsilon}(\boldsymbol{m},(t))| \leqslant C\epsilon|\boldsymbol{m}|^3 \left[\frac{\|\Psi^1\|_{L^2}}{|\boldsymbol{m}|\epsilon} + \frac{\|\Psi^2\|_{L^2}}{|\boldsymbol{m}|^2\epsilon^2} \right] \leqslant C(\boldsymbol{C}_1 + \boldsymbol{C}_2)\epsilon|\boldsymbol{m}|^3|a^{\epsilon}(t)|.$$

Lemma 6. *Suppose* $\Psi_l^{\epsilon}(t,x) = a^{\epsilon}(t)\exp(\mathrm{i}\boldsymbol{m}\cdot x)$*,* Ψ^1 *satisfies (A.1),* Ψ^2 *satisfies (A.2). Denote*

$$g^{\epsilon}(t,x) = \frac{Re^2}{4}\Delta^{-1}(\nabla_2^2 - \nabla_1^2)[\delta((\nabla_1\Psi_l^{\epsilon})^2 + (\nabla_2\Psi_l^{\epsilon})^2) + (1+\delta^2)\nabla_1\Psi_l^{\epsilon}\nabla_2\Psi_l^{\epsilon}],$$

$$f^{\epsilon}(t,x) = M^{\epsilon}\Psi^2.$$

If

$$|\hat{f}^{\epsilon}(t,\boldsymbol{m})| \leqslant \boldsymbol{C}_1|\boldsymbol{m}|^2|a^{\epsilon}(t)|, \qquad |\hat{g}^{\epsilon}(t,\boldsymbol{m})| \leqslant \boldsymbol{C}_2|\boldsymbol{m}|^2|a^{\epsilon}(t)|, \tag{A.9}$$

then

$$|\hat{f}^{\epsilon}(t,\boldsymbol{m}) - \hat{g}^{\epsilon}(t,\boldsymbol{m})| \leqslant C(\boldsymbol{C}_1 + \boldsymbol{C}_2)\epsilon|\boldsymbol{m}|^3|a^{\epsilon}(t)|.$$

Lemma 7. *Suppose* $\Psi_l^{\epsilon}(t,x) = a^{\epsilon}(t)\exp(\mathrm{i}\boldsymbol{m}\cdot x)$*,* Ψ^1 *satisfies (A.1). Denote*

$$g^{\epsilon}(t,x) = -\frac{Re^2}{8}\Delta^{-1}(\nabla_2^2 - \nabla_1^2)[\delta((\nabla_1\Psi_l^{\epsilon})^2 + (\nabla_2\Psi_l^{\epsilon})^2) + (1+\delta^2)\nabla_1\Psi_l^{\epsilon}\nabla_2\Psi_l^{\epsilon}],$$

$$f^{\epsilon}(t,x) = \mathbb{N}^{\epsilon}(\Psi^1).$$

If

$$|\hat{f}^{\epsilon}(t,\boldsymbol{m})| \leqslant \boldsymbol{C}_1|\boldsymbol{m}|^2|a^{\epsilon}(t)|, \qquad |\hat{g}^{\epsilon}(t,\boldsymbol{m})| \leqslant \boldsymbol{C}_2|\boldsymbol{m}|^2|a^{\epsilon}(t)|, \tag{A.10}$$

then

$$|\hat{f}^{\epsilon}(t,\boldsymbol{m}) - \hat{g}^{\epsilon}(t,\boldsymbol{m})| \leqslant C(\boldsymbol{C}_1 + \boldsymbol{C}_2)\epsilon|\boldsymbol{m}|^3|a^{\epsilon}(t)|.$$

Appendix B. Estimates for nonlinear terms

In this appendix we prove auxiliary estimates on the nonlinear terms like

$$\Upsilon = \mathbb{N}^\epsilon(\Psi, \Phi) = \Delta^{-1} J_{xx}(\Psi, \Delta\Phi), \tag{B.1}$$

where the Fourier coefficients of Ψ and Φ have bounds that differ from lemma to lemma. We use the following notation: the Fourier coefficients Ψ (the first function), Φ (the second function) and Υ are indexed by $\boldsymbol{k}_1 = (\boldsymbol{m}_1, \boldsymbol{n}_1)_\epsilon$, $\boldsymbol{k}_2 = (\boldsymbol{m}_2, \boldsymbol{n}_2)_\epsilon$, and $\boldsymbol{k} = (\boldsymbol{m}, \boldsymbol{n})_\epsilon$, respectively. Without loss of generality, we assume here that all the functions are independent of time. The main equality used in all the lemmas below is

$$\begin{aligned}\Upsilon &= J_{xx}(\Psi, \Delta\Phi) = -\nabla_2(\Psi\nabla_1\Delta\Phi) + \nabla_1(\Psi\nabla_2\Delta\Phi)\\ &= -\nabla_1(\Delta\Phi\nabla_2\Psi) + \nabla_2(\Delta\Phi\nabla_1\Psi) = -\nabla_2\Psi\nabla_1\Delta\Phi + \nabla_1\Psi\nabla_2\Delta\Phi.\end{aligned}$$

In terms of Fourier coefficients it gives rise to three inequalities

$$\begin{aligned}|\hat{\Upsilon}(\boldsymbol{k})| &\leqslant |\boldsymbol{k}| \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2} |\boldsymbol{k}_2|^3 |\hat{\Psi}(\boldsymbol{k}_1)||\hat{\Phi}(\boldsymbol{k}_2)|,\\ |\hat{\Upsilon}(\boldsymbol{k})| &\leqslant |\boldsymbol{k}| \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2} |\boldsymbol{k}_1||\boldsymbol{k}_2|^2 |\hat{\Psi}(\boldsymbol{k}_1)||\hat{\Phi}(\boldsymbol{k}_2)|,\\ |\hat{\Upsilon}(\boldsymbol{k})| &\leqslant \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2} |\boldsymbol{k}_1||\boldsymbol{k}_2|^3 |\hat{\Psi}(\boldsymbol{k}_1)||\hat{\Phi}(\boldsymbol{k}_2)|.\end{aligned} \tag{B.2}$$

Depending on the decay of $|\hat{\Psi}(\boldsymbol{k})|$ and $|\hat{\Phi}(\boldsymbol{k})|$ as $|\boldsymbol{k}| \to \infty$, we use one of the three inequalities in (B.2), and then use Hölder's inequality:

$$\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2} \frac{1}{|\boldsymbol{k}_1|^s} \frac{1}{|\boldsymbol{k}_1|^t} \leqslant C(s, p)C(t, q), \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

where we denote

$$C(s, p) = \left(\sum_{\boldsymbol{k}} \frac{1}{|\boldsymbol{k}|^{sp}}\right)^{1/p}. \tag{B.3}$$

Since in two dimensions the right-hand side of (B.3) is finite if $sp > 2$, we choose p and q so that $sp > 2$ and $tq > 2$. By the choice of the σ/ϵ neighbourhoods of the vertices $\boldsymbol{k} = \boldsymbol{n}/\epsilon$, we have that if $\boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2$, $\boldsymbol{k} = (\boldsymbol{m}, \boldsymbol{n})_\epsilon$, $\boldsymbol{k}_1 = (\boldsymbol{m}_1, \boldsymbol{n}_1)_\epsilon$, $\boldsymbol{k}_2 = (\boldsymbol{m}_2, \boldsymbol{n}_2)_\epsilon$, $|\boldsymbol{m}_1| \leqslant \sigma/\epsilon$, $|\boldsymbol{m}_2| \leqslant \sigma/\epsilon$, then $\boldsymbol{m} = \boldsymbol{m}_1 + \boldsymbol{m}_2$, $\boldsymbol{n} = \boldsymbol{n}_1 + \boldsymbol{n}_2$. The inequality $|\boldsymbol{m}| \leqslant \sigma/\epsilon$ may not hold, but if $|\boldsymbol{m}| > \sigma/\epsilon$ and $\boldsymbol{n} = (0, 0)$, or $\boldsymbol{n} = (\pm 1, \pm 1)$, then $|\boldsymbol{k}| \leqslant C|\boldsymbol{m}|$, $C = C(\sigma)$. We also use the observation

$$\begin{aligned}&J_{xx}(\exp(\mathrm{i}\boldsymbol{k}_1 \cdot x), \Delta \exp(\mathrm{i}\boldsymbol{k}_2 \cdot x)) + J_{xx}(\exp(\mathrm{i}\boldsymbol{k}_2 \cdot x), \Delta \exp(\mathrm{i}\boldsymbol{k}_1 \cdot x))\\ &\qquad = (|\boldsymbol{k}_1|^2 - |\boldsymbol{k}_2|^2) J_{xx}(\exp(\mathrm{i}\boldsymbol{k}_1 \cdot x), \exp(\mathrm{i}\boldsymbol{k}_1 \cdot x))\end{aligned}$$

for any $\boldsymbol{k}_1, \boldsymbol{k}_2$. If $\boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2$, then $||\boldsymbol{k}_1|^2 - |\boldsymbol{k}_2|^2| \leqslant C|\boldsymbol{k}| \max(|\boldsymbol{k}_1|, |\boldsymbol{k}_2|)$.

Lemma 8. *Suppose* $\Upsilon = \mathbb{N}(\Psi) - \mathbb{N}(\mathbb{P}^\epsilon_{\mathrm{large}}\Psi)$, $|\hat{\Psi}(\boldsymbol{k})| \leqslant \boldsymbol{C}_0/|\boldsymbol{k}|^4$, *then* $|\Upsilon(\boldsymbol{k})| \leqslant \epsilon \boldsymbol{C}\boldsymbol{C}_0^2/|\boldsymbol{k}|$.

Proof. Let $\Phi(x) = [\mathbb{P}^\epsilon_{\mathrm{small}} + \mathbb{P}^\epsilon_{\mathrm{rest}}]\Psi(x)$, then $|\hat{\Phi}(\boldsymbol{k})| \leqslant \epsilon \boldsymbol{C}\boldsymbol{C}_0/|\boldsymbol{k}|^3$, and $\mathbb{N}(\Psi) - \mathbb{N}(\mathbb{P}^\epsilon_{\mathrm{large}}\Psi) = \mathbb{N}(\Psi, \Phi) + \mathbb{N}(\Phi, \mathbb{P}^\epsilon_{\mathrm{large}}\Psi)$. Therefore

$$\Upsilon(x) = \mathbb{N}^\epsilon(\Psi, \Phi) + \frac{Re^2}{8}\mathbb{D}(\Psi, \Phi) + \mathbb{N}^\epsilon(\Phi, \mathbb{P}^\epsilon_{\mathrm{large}}\Psi) + \frac{Re^2}{8}\mathbb{D}(\Phi, \mathbb{P}^\epsilon_{\mathrm{large}}\Psi),$$

$$\mathbb{D}(\Psi, \Phi) = \Delta^{-1}(\nabla_2^2 - \nabla_1^2)[\delta(\nabla_1\Psi\nabla_1\Phi + \nabla_2\Psi\nabla_2\Phi) + (1 + \delta^2)\nabla_1\Psi\nabla_2\Phi].$$

By the Leibniz rule

$$\mathbb{D}(\Psi,\Phi) = \Delta^{-1}(\nabla_2 - \nabla_1)\big[\delta(\nabla_1^2\Psi\nabla_1\Phi + \nabla_1\Psi\nabla_1^2\Phi + \nabla_1\nabla_2\Psi\nabla_1\Phi + \nabla_1\Psi\nabla_1\nabla_2\Phi + \nabla_2\Psi\nabla_2^2\Phi + \nabla_2^2\Psi\nabla_2\Phi + \nabla_1\nabla_2\Psi\nabla_2\Phi + \nabla_2\Psi\nabla_1\nabla_2\Phi) + (1+\delta^2)(\nabla_1\nabla_2\Psi\nabla_2\Phi + \nabla_1^2\Psi\nabla_2\Phi + \nabla_1\Psi\nabla_2^2\Phi + \nabla_1\Psi\nabla_1\nabla_2\Phi)\big].$$

Therefore

$$\begin{aligned}|\Upsilon(\boldsymbol{k})| &\leqslant \frac{1}{|\boldsymbol{k}|}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3}\frac{\epsilon C\boldsymbol{C}_0}{|\boldsymbol{k}_2|} + \frac{C}{|\boldsymbol{k}|}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\left[\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^2}\frac{\epsilon C\boldsymbol{C}_0}{|\boldsymbol{k}_2|^2} + \frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3}\frac{\epsilon C\boldsymbol{C}_0}{|\boldsymbol{k}_2|}\right]\\ &\quad + \frac{1}{|\boldsymbol{k}|}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\frac{\epsilon C\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_2|} + \frac{1}{|\boldsymbol{k}|}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\left[\frac{\epsilon C\boldsymbol{C}_0}{|\boldsymbol{k}_1|}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_2|^3} + \frac{\epsilon C\boldsymbol{C}_0}{|\boldsymbol{k}_2|^2}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^2}\right]\\ &\leqslant \frac{\epsilon C\boldsymbol{C}_0^2}{|\boldsymbol{k}|}\left((C(2,2))^2 + C(1,3)C\left(3,\frac{3}{2}\right)\right) \leqslant \frac{\epsilon C\boldsymbol{C}_0^2}{|\boldsymbol{k}|}.\end{aligned}$$

■

Lemma 9. *Suppose* $\Upsilon = \mathbb{N}(\Psi_1,\Phi) + \mathbb{N}(\Phi,\Psi_2)$. *If* $|\hat{\Psi}_1(\boldsymbol{k})| \leqslant \boldsymbol{C}_1/|\boldsymbol{k}|^4$, $|\hat{\Psi}_2(\boldsymbol{k})| \leqslant \boldsymbol{C}_1/|\boldsymbol{k}|^4$, $|\hat{\Phi}(\boldsymbol{k})| \leqslant \boldsymbol{C}_0/|\boldsymbol{k}|^2$, *then* $|\hat{\Upsilon}(\boldsymbol{k})| \leqslant C\boldsymbol{C}_1\boldsymbol{C}_0$.

Proof. Here we *do not* apply the Leibniz rule.

$$\begin{aligned}|\hat{\Upsilon}(\boldsymbol{k})| &\leqslant \frac{1}{|\boldsymbol{k}|}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\frac{\boldsymbol{C}_1}{|\boldsymbol{k}_1|^3}\boldsymbol{C}_0 + \frac{C}{|\boldsymbol{k}|}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|}\frac{\boldsymbol{C}_1}{|\boldsymbol{k}_2|^2} + C\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|}\frac{\boldsymbol{C}_1}{|\boldsymbol{k}_2|^3}\\ &\leqslant C\boldsymbol{C}_1\boldsymbol{C}_0\left(C(3,1) + C(1,\tfrac{5}{2})C(2,\tfrac{5}{3}) + C(1,3)C(3,\tfrac{3}{2})\right) \leqslant C\boldsymbol{C}_1\boldsymbol{C}_0.\end{aligned}$$

■

Lemma 10. *Suppose* $\Upsilon = \mathbb{P}^{\epsilon}_{\mathrm{small}}\mathbb{N}^{\epsilon}(\Psi,\Phi)$. $\Psi = \mathbb{P}^{\epsilon}_{\mathrm{large}}\Psi$, $\Phi = \mathbb{P}^{\epsilon}_{\mathrm{middle}}\Phi$. *If*

$$|\hat{\Psi}(\boldsymbol{k})| \leqslant \frac{C}{|\boldsymbol{k}|^2}, \qquad |\hat{\Phi}(\boldsymbol{k})| \leqslant \frac{C}{|\boldsymbol{k}||\boldsymbol{m}|^3} \tag{B.4}$$

or

$$|\hat{\Psi}(\boldsymbol{k})| \leqslant \frac{C}{|\boldsymbol{k}|^4}, \qquad |\hat{\Phi}(\boldsymbol{k})| \leqslant \frac{C}{|\boldsymbol{k}||\boldsymbol{m}|}, \tag{B.5}$$

then $|\hat{\Upsilon}(\boldsymbol{k})| \leqslant C$.

Proof. The only nonzero Fourier coefficients $\hat{\Phi}(\boldsymbol{k})$ are when $\boldsymbol{n} = (\pm 1, \pm 1)$. If (B.4) holds then

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \frac{C}{|\boldsymbol{k}|^2}\sum_{\boldsymbol{m}=\boldsymbol{m}_1+\boldsymbol{m}_2}\frac{1}{|\boldsymbol{m}_1|}\frac{|\boldsymbol{k}_2|^2}{|\boldsymbol{m}_2|^3} \leqslant CC(1,3)C\left(3,\frac{3}{2}\right) \leqslant C,$$

because $|\boldsymbol{k}_2| \leqslant C|\boldsymbol{k}|$. If (B.5) holds then, similarly,

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \frac{C}{|\boldsymbol{k}|^2}\sum_{\boldsymbol{m}=\boldsymbol{m}_1+\boldsymbol{m}_2}\frac{1}{|\boldsymbol{m}_1|^3}\frac{|\boldsymbol{k}_2|^2}{|\boldsymbol{m}_2|} \leqslant CC\left(3,\frac{3}{2}\right)C(1,3) \leqslant C.$$

■

Lemma 11. *Suppose* $\Upsilon = \mathbb{N}^{\epsilon}(\Psi)$, $|\hat{\Psi}(\boldsymbol{k})| \leqslant \boldsymbol{C}_0/|\boldsymbol{k}|^4$, *then* $|\Upsilon(\boldsymbol{k})| \leqslant C\boldsymbol{C}_0^2/|\boldsymbol{k}|^2$.

Proof.

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant C\frac{1}{|\boldsymbol{k}|^2}\sum_{k=k_1+k_2}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_2|} \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}C\left(3,\frac{3}{2}\right)C(1,3) \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}. \qquad \blacksquare$$

In the rest of this section we assume without further mention (cf definition 3 of lacunary spaces and inequality (3.11) in section (3.2)) that

$$|\hat{\Psi}(\boldsymbol{k})| \leqslant \frac{\boldsymbol{C}_0}{|\boldsymbol{k}|^4},$$

$$|\hat{\Xi}^\epsilon(\boldsymbol{k})| \leqslant \begin{cases} \dfrac{\boldsymbol{C}_0}{2|\boldsymbol{k}||\boldsymbol{m}|^3}, & \boldsymbol{n} = (\pm1,\pm1), \quad |\boldsymbol{m}| < \dfrac{\sigma}{\epsilon}, \\ 0, & \text{otherwise}, \end{cases}$$

$$|\hat{\Theta}_1^\epsilon(\boldsymbol{k})| \leqslant \begin{cases} \dfrac{\boldsymbol{C}_0}{2^{\|\boldsymbol{n}\|}|\boldsymbol{k}|^2|\boldsymbol{m}|^2}, & \boldsymbol{n} \neq (0,0), \quad |\boldsymbol{m}| < \dfrac{\sigma}{\epsilon}, \\ 0, & \text{otherwise}, \end{cases}$$

$$|\hat{\Theta}_2^\epsilon(\boldsymbol{k})| \leqslant \begin{cases} \dfrac{\boldsymbol{C}_0}{|\boldsymbol{m}||\boldsymbol{k}|^3}, & \boldsymbol{n} \neq (0,0), \quad \boldsymbol{m}| < \dfrac{\sigma}{\epsilon}, \\ 0, & \text{otherwise}. \end{cases}$$

Lemma 12. *Suppose* $\Upsilon = \mathbb{N}^\epsilon(\Xi^\epsilon)$*, then* $|\hat{\Upsilon}(\boldsymbol{k})| \leqslant C\boldsymbol{C}_0^2/(|\boldsymbol{k}||\boldsymbol{m}|)$.

Proof. Note that $\hat{\Upsilon}(\boldsymbol{k}) \equiv 0$, if $\boldsymbol{n} \neq (\pm1,\pm1) + (\pm1,\pm1)$.

$$||\boldsymbol{k}_2|^2 - |\boldsymbol{k}_1|^2| = \left|\frac{(\pm1,\pm1)}{\epsilon} + \boldsymbol{m}_2\right|^2 - \left|\frac{(\pm1,\pm1)}{\epsilon} + \boldsymbol{m}_1\right|^2$$
$$\leqslant \left|\left(\frac{2}{\epsilon^2} + \frac{C}{\epsilon}|\boldsymbol{m}_2| + |\boldsymbol{m}_2|^2 - \frac{2}{\epsilon^2} + \frac{C}{\epsilon}|\boldsymbol{m}_1| + |\boldsymbol{m}_1|^2\right)\right| \leqslant C\frac{|\boldsymbol{m}_1| + |\boldsymbol{m}_2|}{\epsilon}.$$

Also $1/\epsilon \leqslant C\max(|\boldsymbol{k}_1|,|\boldsymbol{k}_2|)$, therefore

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|}\sum_{k=k_1+k_2,|k_1|\leqslant|k_2|}||\boldsymbol{k}_2|^2 - |\boldsymbol{k}_1|^2|\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_1|^3}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_2||\boldsymbol{m}_2|^3}$$
$$\leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|}\sum_{m=m_1+m_2}\frac{|\boldsymbol{m}_1| + |\boldsymbol{m}_2|}{|\boldsymbol{m}_1|^3|\boldsymbol{m}_2|^3} \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|}\sum_{m=m_1+m_2,|m_1|>|m_2|}\frac{(|\boldsymbol{m}_1| + |\boldsymbol{m}_2|)^2}{|\boldsymbol{m}_1|^3|\boldsymbol{m}_2|^3}$$
$$\leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|}\left(C(1,3)C\left(3,\frac{3}{2}\right) + (C(2,2))^2\right) \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|}. \qquad \blacksquare$$

Lemma 13. *Suppose* $\Upsilon = \mathbb{N}^\epsilon(\Psi,\Xi^\epsilon)$*, there is a sufficiently small* ϵ_0 *such that for any* $\epsilon < \epsilon_0$

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \begin{cases} C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{m}|^2}, & \boldsymbol{n} = (\pm1,\pm1), \quad |\boldsymbol{m}| \leqslant \dfrac{\sigma}{\epsilon}, \\ \epsilon C\dfrac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}, & \text{otherwise}. \end{cases}$$

Proof.

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|^2} \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2, \boldsymbol{n}_2=(\pm1,\pm1)} \frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3} \frac{\boldsymbol{C}_0|\boldsymbol{k}_2|^2}{|\boldsymbol{m}_2|^3}.$$

If $\boldsymbol{n} = (\pm1, \pm1)$, $|\boldsymbol{m}| \leqslant \sigma/\epsilon$, then $|\boldsymbol{k}| \leqslant C|\boldsymbol{k}_2|$. Also for sufficiently small ϵ inequality $|\boldsymbol{k}_1|^3 \leqslant |\boldsymbol{k}_2|^2$ implies that $\boldsymbol{n} = \boldsymbol{n}_2$. Therefore

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|^2}\left[\sum_{\substack{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2,\\ |\boldsymbol{k}_1|^3>|\boldsymbol{k}_2|^2}} + \sum_{\substack{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2,\\ |\boldsymbol{m}_2|^3>|\boldsymbol{k}_2|^2}} + \sum_{\substack{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2,\\ |\boldsymbol{k}_1|^3\leqslant|\boldsymbol{k}_2|^2,\\ |\boldsymbol{m}_2|^3\leqslant|\boldsymbol{k}_2|^2}}\right] \frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3} \frac{\boldsymbol{C}_0|\boldsymbol{k}_2|^2}{|\boldsymbol{m}_2|^3}$$

$$\leqslant \frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}\left[\sum_{\boldsymbol{m}_2} \frac{1}{|\boldsymbol{m}_2|^3} + \sum_{\boldsymbol{k}_1} \frac{1}{|\boldsymbol{k}_1|^3} + C|\boldsymbol{k}|^2 \sum_{\substack{\boldsymbol{m}_1=\boldsymbol{k}_1,\, \boldsymbol{n}=\boldsymbol{n}_2,\\ \boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}} \frac{1}{|\boldsymbol{k}_1|^3} \frac{1}{|\boldsymbol{m}_2|^3}\right]$$

$$\leqslant 2C(3,1)\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2} + C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{m}|^2} \sum_{\boldsymbol{m}=\boldsymbol{m}_1+\boldsymbol{m}_2} \frac{(|\boldsymbol{m}_1|+|\boldsymbol{m}_2|)^2}{|\boldsymbol{m}_1|^3|\boldsymbol{m}_2|^3}$$

$$\leqslant C\left(C(1,3)C\left(3,\frac{3}{2}\right) + (C(2,2))^2\right)\frac{\boldsymbol{C}_0^2}{|\boldsymbol{m}|^2} \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{m}|^2}.$$

In all other cases $|\boldsymbol{k}_1| > C/\epsilon$ and $|\boldsymbol{k}_2| \leqslant C|\boldsymbol{k}_1|$, therefore

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|^2} \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2} \frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3} \frac{\boldsymbol{C}_0|\boldsymbol{k}_2|^2}{|\boldsymbol{m}_2|^3} \leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2} \sum \frac{1}{|\boldsymbol{m}_2|^3} \leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}. \qquad \blacksquare$$

Lemma 14. *Suppose* $\Upsilon = \mathbb{N}^\epsilon(\Psi, \Theta_1^\epsilon)$, *then*

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \begin{cases} C\dfrac{\boldsymbol{C}_0^2}{|\boldsymbol{m}||\boldsymbol{k}|}, & \boldsymbol{n} = (\pm1,\pm1), \quad |\boldsymbol{m}| \leqslant \dfrac{\sigma}{\epsilon}, \\ \epsilon C\dfrac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}, & \text{otherwise.} \end{cases}$$

Proof.

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|^2} \sum_{\substack{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2,\\ \boldsymbol{n}_2\neq(0,0)}} \frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3} \frac{\boldsymbol{C}_0|\boldsymbol{k}_2|}{2^{\|\boldsymbol{n}_2\|}|\boldsymbol{m}_2|^2} \leqslant \frac{1}{\epsilon|\boldsymbol{k}|^2} \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2} \frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3} \frac{\boldsymbol{C}_0\|\boldsymbol{n}_2\|}{2^{\|\boldsymbol{n}_2\|}|\boldsymbol{m}_2|^2}, \tag{B.6}$$

because $\boldsymbol{n}_2 \neq (0,0)$, and therefore $|\boldsymbol{k}_2| \leqslant C\|\boldsymbol{n}_2\|/\epsilon$.

If $\boldsymbol{n} = (\pm1,\pm1)$ and $|\boldsymbol{m}| \leqslant \sigma/\epsilon$ is *not* true, then $|\boldsymbol{k}_1| > C/\epsilon$ and

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \frac{1}{\epsilon|\boldsymbol{k}|^2} \sum_{\substack{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2,\\ |\boldsymbol{k}_1|>C/\epsilon}} \frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3} \frac{\boldsymbol{C}_0\|\boldsymbol{n}_2\|}{2^{\|\boldsymbol{n}_2\|}|\boldsymbol{m}_2|^2} \leqslant C\frac{\boldsymbol{C}_0^2}{\epsilon|\boldsymbol{k}|^2}\left(\sum_{|\boldsymbol{k}_1|>C/\epsilon} \frac{1}{|\boldsymbol{k}_1|^6}\right)^{1/2}$$

$$\times\left(\sum_{\boldsymbol{m}_2}\sum_{\boldsymbol{n}_2} \frac{\|\boldsymbol{n}_2\|^2}{4^{\|\boldsymbol{n}_2\|}|\boldsymbol{m}_2|^4}\right)^{1/2} \leqslant C\frac{\boldsymbol{C}_0^2}{\epsilon|\boldsymbol{k}|^2}\epsilon^2\left(\sum_{\boldsymbol{m}_2} \frac{1}{|\boldsymbol{m}_2|^4}\right)^{1/2} \leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}.$$

If $\boldsymbol{n} = (\pm1,\pm1)$ and $|\boldsymbol{m}| \leqslant \sigma/\epsilon$, then $|\boldsymbol{k}| \geqslant C/\epsilon$, and in the last sum in (B.6) either $\|\boldsymbol{n}_2\| = 1$ (i.e. $\boldsymbol{n}_2 = (\pm1,\pm1)$), or $\|\boldsymbol{n}_2\| > 1$, $|\boldsymbol{k}_1| > C/\epsilon$. Since

$$|\boldsymbol{m}| \leqslant 2\max(|\boldsymbol{m}_1|, |\boldsymbol{m}_2|) \leqslant 2\max(|\boldsymbol{k}_1|, |\boldsymbol{m}_2|),$$

we have

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \frac{1}{\epsilon|\boldsymbol{k}|^2}\left[\sum_{\substack{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2,\\ \|\boldsymbol{n}_2\|=1}} + \sum_{\substack{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2,\\ \|\boldsymbol{n}_2\|>1,\\ |\boldsymbol{k}_1|>C/\epsilon}}\right]\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3}\frac{\boldsymbol{C}_0\|\boldsymbol{n}_2\|}{2^{\|\boldsymbol{n}_2\|}|\boldsymbol{m}_2|^2} \leqslant C\frac{1}{|\boldsymbol{k}|}\sum_{\substack{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2,\\ \|\boldsymbol{n}_2\|=1}}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3}\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_2|^2}$$

$$+\epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2} \leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2} + \frac{1}{|\boldsymbol{k}|}\sum_{\substack{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2,\\ \|\boldsymbol{n}_2\|=1}}\frac{2\max(|\boldsymbol{k}_1|,|\boldsymbol{m}_2|)}{|\boldsymbol{m}|}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3}\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_2|^2}$$

$$\leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2} + C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{m}||\boldsymbol{k}|}\left((C(2,2))^2 + C\left(3,\frac{3}{2}\right)C(1,3)\right)$$

$$\leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2} + C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{m}||\boldsymbol{k}|} \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{m}||\boldsymbol{k}|}. \qquad \blacksquare$$

Lemma 15. *Suppose* $\Upsilon_1 = \mathbb{N}^\epsilon(\Xi^\epsilon,\Theta_1^\epsilon) + \mathbb{N}^\epsilon(\Theta_1^\epsilon,\Xi^\epsilon)$, $\Upsilon_2 = \mathbb{N}^\epsilon(\Theta_1^\epsilon)$. *Then for* $i = 1, 2$

$$|\hat{\Upsilon}_i(\boldsymbol{k})| \leqslant \begin{cases} \epsilon C\dfrac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}, & \text{if } \boldsymbol{n}=(0,0),\\[2ex] C\dfrac{\boldsymbol{C}_0^2}{|\boldsymbol{m}||\boldsymbol{k}|}, & \text{otherwise.}\end{cases}$$

Proof.

$$|\hat{\Upsilon}_1(\boldsymbol{k})| \leqslant C\frac{1}{|\boldsymbol{k}|^2}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2,\boldsymbol{n}_1=(\pm1,\pm1)}||\boldsymbol{k}_1|^2-|\boldsymbol{k}_2|^2|\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_1|^3}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_2||\boldsymbol{m}_2|^2}.$$

Since $|\boldsymbol{k}_1| \leqslant C|\boldsymbol{k}_2|$, we have $||\boldsymbol{k}_1|^2-|\boldsymbol{k}_2|^2| \leqslant C|\boldsymbol{k}_2||\boldsymbol{k}|$. Since $\boldsymbol{m}=\boldsymbol{m}_1+\boldsymbol{m}_2$,

$$|\hat{\Upsilon}_1(\boldsymbol{k})| \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|}\sum_{\boldsymbol{m}=\boldsymbol{m}_1+\boldsymbol{m}_2}\frac{(|\boldsymbol{m}_1|+|\boldsymbol{m}_2|)}{|\boldsymbol{m}_1|^3|\boldsymbol{m}_2|^2}$$

$$\leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|}\left((C(2,2))^2 + C\left(3,\frac{3}{2}\right)C(1,3)\right) \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|}.$$

If $\boldsymbol{n}=(0,0)$ then $\boldsymbol{n}_2=(\pm1,\pm1)$ and $|\boldsymbol{k}| \leqslant C\max(|\boldsymbol{m}_1|,|\boldsymbol{m}_2|)$. Therefore

$$|\hat{\Upsilon}_1(\boldsymbol{k})| \leqslant C\frac{1}{|\boldsymbol{k}|}\sum_{\substack{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2,\\ |\boldsymbol{m}_1|\leqslant|\boldsymbol{m}_2|}}||\boldsymbol{k}_1|^2-|\boldsymbol{k}_2|^2|\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1||\boldsymbol{m}_1|^3}\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_2|^2|\boldsymbol{k}_2|}$$

$$\leqslant \epsilon C\boldsymbol{C}_0^2\sum_{\boldsymbol{m}=\boldsymbol{m}_1+\boldsymbol{m}_2,|\boldsymbol{m}_1|\leqslant|\boldsymbol{m}_2|}\frac{1}{|\boldsymbol{m}_1|^3|\boldsymbol{m}_2|^2} \leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}C(3,1) \leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}.$$

Similarly

$$|\hat{\Upsilon}_2(\boldsymbol{k})| \leqslant C\frac{1}{|\boldsymbol{k}|^2}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}||\boldsymbol{k}_1|^2-|\boldsymbol{k}_2|^2|\frac{\boldsymbol{C}_0}{2^{\|\boldsymbol{n}_1\|}|\boldsymbol{k}_1||\boldsymbol{m}_1|^2}\frac{\boldsymbol{C}_0}{2^{\|\boldsymbol{n}_2\|}|\boldsymbol{k}_2||\boldsymbol{m}_2|^2} \leqslant \frac{\epsilon C\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|}$$

$$\times\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\frac{|\boldsymbol{m}_1|+|\boldsymbol{m}_2|}{2^{\|\boldsymbol{n}_1\|+\|\boldsymbol{n}_2\|}|\boldsymbol{m}_1|^2||\boldsymbol{m}_2|^2} \leqslant \frac{\epsilon C\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|}C\left(1,\frac{5}{2}\right)C\left(2,\frac{5}{3}\right) \leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|},$$

which immediately gives the required estimates ($\boldsymbol{k}=\boldsymbol{m}$ for $\boldsymbol{n}=(0,0)$). $\blacksquare$

Lemma 16. *Suppose* $\Upsilon = \mathbb{N}^\epsilon(\Theta_2^\epsilon)$, *then* $|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \epsilon C \boldsymbol{C}_0^2/(|\boldsymbol{m}||\boldsymbol{k}|)$.

Proof.

$$|\hat{\Upsilon}(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|} \sum_{\substack{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2,\\ C/\epsilon \leqslant |\boldsymbol{k}_1| \leqslant |\boldsymbol{k}_2|}} \frac{\boldsymbol{C}_0^2||\boldsymbol{k}_2|^2 - |\boldsymbol{k}_1|^2|}{|\boldsymbol{m}_1||\boldsymbol{k}_1|^3|\boldsymbol{m}_2|^2|\boldsymbol{k}_2|^2}$$

$$\leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\frac{|\boldsymbol{m}|}{|\boldsymbol{m}_1||\boldsymbol{k}_1|^3|\boldsymbol{m}_2|} \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|}\sum_{|\boldsymbol{k}_1|>C/\epsilon}\frac{1}{|\boldsymbol{k}_1|^3} \leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{m}||\boldsymbol{k}|}. \qquad \blacksquare$$

Lemma 17. *Suppose* $\Upsilon_1 = \mathbb{N}^\epsilon(\Xi^\epsilon, \Psi)$, $\Upsilon_2 = \mathbb{N}^\epsilon(\Xi^\epsilon, \Theta_2^\epsilon) + \mathbb{N}^\epsilon(\Theta_2^\epsilon, \Xi^\epsilon)$, $\Upsilon_3 = \mathbb{N}^\epsilon(\Theta_1^\epsilon, \Psi)$, $\Upsilon_4 = \mathbb{N}^\epsilon(\Psi, \Theta_2^\epsilon) + \mathbb{N}^\epsilon(\Theta_2^\epsilon, \Psi)$, $\Upsilon_5 = \mathbb{N}^\epsilon(\Theta_1^\epsilon, \Theta_2^\epsilon) + \mathbb{N}^\epsilon(\Theta_2^\epsilon, \Theta_1^\epsilon)$. *Then for every* $i = 1, \ldots, 5$

$$|\hat{\Upsilon}_i(\boldsymbol{k})| \leqslant \begin{cases} C\dfrac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}, & |\boldsymbol{k}| > \dfrac{\sigma}{\epsilon}, \\[2ex] \epsilon C\dfrac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}, & |\boldsymbol{k}| \leqslant \dfrac{\sigma}{\epsilon}. \end{cases}$$

Proof.

$$|\hat{\Upsilon}_1(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|^2}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_1|^3}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_2|} \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}C\left(3, \frac{3}{2}\right)C(1,3) \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}.$$

If $|\boldsymbol{k}| \leqslant \sigma/\epsilon$, then $|\boldsymbol{k}_2| > \sigma/\epsilon$, hence

$$|\hat{\Upsilon}_1(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|^2}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_1|^3}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_2|} \leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}C(3,1) \leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}.$$

$$|\hat{\Upsilon}_2(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|^2}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}||\boldsymbol{k}_2|^2 - |\boldsymbol{k}_1|^2|\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_1|^3}\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_2||\boldsymbol{k}_2|^2}.$$

Since $\hat{\Xi}^\epsilon(\boldsymbol{k}_1) \neq 0$ only if $\boldsymbol{n}_1 = (\pm1, \pm1)$, therefore $\boldsymbol{n}_2 = -\boldsymbol{n} + (\pm1, \pm1)$, $||\boldsymbol{k}_2|^2 - |\boldsymbol{k}_1|^2| \leqslant C|\boldsymbol{k}_2|^2$ and we have

$$|\hat{\Upsilon}_2(\boldsymbol{k})| \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}\sum_{\boldsymbol{m}=\boldsymbol{m}_1+\boldsymbol{m}_2}\frac{1}{|\boldsymbol{m}_1|^3|\boldsymbol{m}_2|} \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}C\left(3, \frac{3}{2}\right)C(1,3) \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}.$$

If $|\boldsymbol{k}| \leqslant \sigma/\epsilon$, then $\boldsymbol{k} = \boldsymbol{m}$. Using $||\boldsymbol{k}_2|^2 - |\boldsymbol{k}_1|^2| \leqslant C|\boldsymbol{k}_2||\boldsymbol{k}|$, $|\boldsymbol{k}_2| \geqslant C/\epsilon$, we have

$$|\hat{\Upsilon}_2(\boldsymbol{k})| \leqslant \frac{C}{|\boldsymbol{k}|}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_1|^3}\frac{\boldsymbol{C}_0}{|\boldsymbol{m}_2||\boldsymbol{k}_2|} \leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|}\sum_{\boldsymbol{m}=\boldsymbol{m}_1+\boldsymbol{m}_2}\frac{|\boldsymbol{m}_1| + |\boldsymbol{m}_2|}{|\boldsymbol{m}_1|^3|\boldsymbol{m}_2|}$$

$$\leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}||\boldsymbol{m}|}\left(C\left(2, \frac{3}{2}\right)C(1,3) + C(3,1)\right) \leqslant \epsilon C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}.$$

$$|\hat{\Upsilon}_3(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|^2}\sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2}\frac{\boldsymbol{C}_0}{2^{\|\boldsymbol{n}_1\|}|\boldsymbol{k}_1||\boldsymbol{m}_1|^2}\frac{\boldsymbol{C}_0}{|\boldsymbol{k}_2|} \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}\sum_{\boldsymbol{m}_1}\frac{1}{|\boldsymbol{m}_1|^3} \leqslant C\frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}.$$

If $|\boldsymbol{k}| \leqslant \sigma/\epsilon$, then $|\boldsymbol{k}_2| > \sigma/\epsilon$, hence

$$|\hat{\Upsilon}_3(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|^2} \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2} \frac{\boldsymbol{C}_0}{2^{\|\boldsymbol{n}_1\|}|\boldsymbol{k}_1||\boldsymbol{m}_1|^2} \frac{\boldsymbol{C}_0}{|\boldsymbol{k}_2|} \leqslant \epsilon C \frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2} \sum_{\boldsymbol{m}_1} \frac{1}{|\boldsymbol{m}_1|^3} \leqslant \epsilon C \frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}.$$

$$|\hat{\Upsilon}_4(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|^2} \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2} ||\boldsymbol{k}_2|^2 - |\boldsymbol{k}_1|^2| \frac{\boldsymbol{C}_0}{|\boldsymbol{k}_1|^3} \frac{\boldsymbol{C}_0}{|\boldsymbol{m}_2||\boldsymbol{k}_2|^2} \leqslant C \frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}$$
$$\times \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2} \left[\frac{1}{|\boldsymbol{k}_1|^3} + \frac{1}{|\boldsymbol{k}_1||\boldsymbol{k}_2|^2} \right] \leqslant C \frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2} \left(C(3,1) + C(1,3) C\left(2, \frac{3}{2}\right) \right) \leqslant C \frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}.$$

If $|\boldsymbol{k}| \leqslant \sigma/\epsilon$, then $|\boldsymbol{k}_1| > \sigma/\epsilon$, hence

$$|\hat{\Upsilon}_4(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|^2} \sum_{|\boldsymbol{k}_1|>\sigma/\epsilon} \frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}_1|^3} \leqslant \epsilon C \frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}.$$

$$|\hat{\Upsilon}_5(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|^2} \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2} ||\boldsymbol{k}_2|^2 - |\boldsymbol{k}_1|^2| \frac{\boldsymbol{C}_0}{2^{\|\boldsymbol{n}_1\|}|\boldsymbol{m}_1|^2|\boldsymbol{k}_1|} \frac{\boldsymbol{C}_0}{|\boldsymbol{m}_2||\boldsymbol{k}_2|^2}$$
$$\leqslant C \frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2} \sum_{\boldsymbol{m}_1} \sum_{\boldsymbol{n}_1} \frac{1}{2^{\|\boldsymbol{n}_1\|}|\boldsymbol{m}_1|^3} \leqslant C \frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}.$$

If $|\boldsymbol{k}| \leqslant \sigma/\epsilon$, then $\boldsymbol{k} = \boldsymbol{m}$ and $C_1 \leqslant |\boldsymbol{k}_2|/|\boldsymbol{k}_1| \leqslant C_2$. Hence

$$|\hat{\Upsilon}_5(\boldsymbol{k})| \leqslant \frac{1}{|\boldsymbol{k}|} \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2} ||\boldsymbol{k}_2|^2 - |\boldsymbol{k}_1|^2| \frac{\boldsymbol{C}_0}{2^{\|\boldsymbol{n}_1\|}|\boldsymbol{m}_1|^2|\boldsymbol{k}_1|^2} \frac{\boldsymbol{C}_0}{|\boldsymbol{m}_2||\boldsymbol{k}_2|^2}$$
$$\leqslant C\boldsymbol{C}_0^2 \sum_{\boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2, |\boldsymbol{m}_2| \leqslant |\boldsymbol{m}_1|} \frac{1}{|\boldsymbol{m}_1|^2|\boldsymbol{k}_1|^2} \frac{1}{|\boldsymbol{m}_2||\boldsymbol{k}_2|^2}$$
$$\leqslant C \frac{\boldsymbol{C}_0^2}{|\boldsymbol{m}|^2} \sum_{\substack{|\boldsymbol{k}_1| \geqslant \sigma/\epsilon, \\ |\boldsymbol{k}_2| \geqslant \sigma/\epsilon}} \left[\frac{1}{|\boldsymbol{k}_1|^2|\boldsymbol{k}_2|} + \frac{1}{|\boldsymbol{k}_1||\boldsymbol{k}_2|^2} \right]$$
$$\leqslant C \frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2} \left(\epsilon C(1,3) C\left(2, \frac{3}{2}\right) \right) \leqslant \epsilon C \frac{\boldsymbol{C}_0^2}{|\boldsymbol{k}|^2}. \qquad \blacksquare$$

As a corollary of lemmas 11–17, we have an embedding result for the nonlinear term for functions in L^ϵ.

Lemma 18. *Suppose* $\|\Psi_1^\epsilon\|_{L^\epsilon} \leqslant \boldsymbol{C}_0$, $\|\Psi_2^\epsilon\|_{L^\epsilon} \leqslant \boldsymbol{C}_0$, $\|\Psi_1^\epsilon - \Psi_2^\epsilon\|_{L^\epsilon} \leqslant \boldsymbol{C}_1$. *Let* $\Psi_3^\epsilon = \Delta^{-1}\mathbb{N}^\epsilon(\Psi_1^\epsilon)$, $\Psi_4^\epsilon = \Delta^{-1}\mathbb{N}^\epsilon(\Psi_2^\epsilon)$, $\Psi_5^\epsilon = \Psi_3 - \Psi_4$. *Then*

(i) $\|\Psi_i^\epsilon\|_{L^\epsilon} \leqslant C\boldsymbol{C}_0^2$, $i = 3, 4$, $\|\Psi_5^\epsilon\|_{L^\epsilon} \leqslant C\boldsymbol{C}_0\boldsymbol{C}_1$.

(ii) If we expand any of Ψ_i^ϵ, $i = 3, 4, 5$ *in the* L^ϵ *lacunary sum* $\Psi_i^\epsilon = \Psi_{i,l}^\epsilon + \Xi_i^\epsilon + \Theta_i^\epsilon + \Psi_{i,r}^\epsilon$, *then* $\Xi_i^\epsilon \equiv 0$.

Similarly, by lemmas 11–13, we have an embedding result for the nonlinear term for functions in L_0^ϵ.

Lemma 19. *Suppose* $\|\Psi_1^\epsilon\|_{L_0^\epsilon} \leqslant \boldsymbol{C}_0$, $\|\Psi_2^\epsilon\|_{L_0^\epsilon} \leqslant \boldsymbol{C}_0$, $\|\Psi_1^\epsilon - \Psi_2^\epsilon\|_{L_0^\epsilon} \leqslant \boldsymbol{C}_1$. *Define* Ψ_3^ϵ *and* Ψ_4^ϵ *as follows: for* $i = 1, 2$ *let*

$$\Psi_{i+2}^\epsilon = \mathbb{P}_{\text{large}}^\epsilon \left[\mathbb{N}^\epsilon(\Psi_{i,l,0}^\epsilon) + \mathbb{N}^\epsilon(\Xi_{i,0}^\epsilon) \right] + \mathbb{P}_{\text{small}}^\epsilon \mathbb{N}^\epsilon(\Psi_{i,l,0}^\epsilon, \Xi_{i,0}^\epsilon).$$

Let $\Psi_5^\epsilon = \Psi_3^\epsilon - \Psi_4^\epsilon$. *Then*

(i) $\|\Psi_i^\epsilon\|_{L_0^\epsilon} \leqslant CC_0^2$, $i = 3, 4$, $\|\Psi_5^\epsilon\|_{L_0^\epsilon} \leqslant CC_0C_1$.

(ii) If we expand any of Ψ_i^ϵ, $i = 3, 4, 5$ *in the* L_0^ϵ *lacunary sum* $\Psi_i^\epsilon = \Psi_{i,l,0}^\epsilon + \Xi_{i,0}^\epsilon + \Theta_{i,0}^\epsilon$, *then* $\Xi_{i,0}^\epsilon \equiv 0$.

References

[1] Allaire G 1992 Homogenization and two-scale convergence *SIAM J. Math. Anal.* **23** 1482–518

[2] Allaire G and Conca C 1996 Bloch-wave homogenization for a spectral problem in fluid-solid structures *Arch. Rat. Mech. Anal.* **135** 197–257

[3] Allaire G, Conca C and Vanninathan M 1998 The Bloch transform and applications. Actes du 29ème Congrès d'Analyse Numérique: CANum'97 (Larnas, 1997) *ESAIM Proc., 3, Soc. Math. Appl. Indust. (Paris)* pp 65–84

[4] Arnold V I and Khesin B A 1998 *Topological Methods in Hydrodynamics* (*Applied Mathematical Sciences* vol 125) (New York: Springer)

[5] Babin A, Mahalov A and Nicolaenko B 2001 3D Navier–Stokes and Euler equations with initial data characterized by uniformly large vorticity *Ind. Univ. Math. J.* **50** 1–35

[6] Babin A, Mahalov A and Nicolaenko B 1996 Global splitting, integrability and regularity of 3D Euler and Navier–Stokes equations for uniformly rotating fluids *Eur. J. Mech.* B/*Fluids* **15** 291–300

[7] Bakhvalov N S and Panasenko G P 1989 *Homogenization: Averaging Processes in Periodic Media* (Dordrecht: Kluwer)

[8] Bensoussan A, Lions J-L and Papanicolaou G 1978 *Asymptotic Analysis for Periodic Structures* (Amsterdam: North-Holland)

[9] Berlyand L V 1983 Averaging of equations of the linear theory of elasticity in domains with a fine-grained boundary I *Teor. Funktsii Funktsional. Anal. Prilozhen.* **39** 16–25

[10] Bloch F 1928 Über die quantenmechanik der electronen in kristallgrttern *Z. Phys.* **52** 555–600

[11] Conca C, Orive R and Vanninathan M 2002 Bloch approximation in homogenization and applications *SIAM J. Math. Anal.* **33** 1166–98

[12] Constantin P and Foias C 1988 *Navier–Stokes Equations* (Chicago, IL: University of Chicago Press)

[13] Constantin P, Foias C, Nicolaenko B and Temam R 1989 *Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations* (*Applied Mathematical Sciences* vol 70) (New York: Springer)

[14] Dando A H, Hall P and Papageorgiou D T 2000 The modulational stability of Taylor vortices in a curved channel *SIAM J. Appl. Math.* **60** 1543–64

[15] De Giorgi E 1975 Sulla convergenza di alcune successioni d'integrali del tipo dell'area *Rend. Mat.* **8** 277–94

[16] Dubrulle B and Frisch U 1991 Eddy viscosity of parity-invariant flows *Phys. Rev.* A **43** 5355–64

[17] Frisch U, She Z S and Sulem P L 1987 Large-scale flow driven by the anisotropic kinetic alpha effect *Physica* D **28** 382–92

[18] Floquet G 1883 Sur les équations différentielles linéaires à coefficients périodiques *Ann. Sci. École Norm. Sup.* **12** 47–88

[19] Friedlander S, Strauss W and Vishik M 1999 Robustness of instability for the two-dimensional Euler equations *SIAM J. Math. Anal.* **30** 1343–54

[20] Gama S, Vergassola M and Frisch U 1994 Negative eddy viscosity in isotropically forced two-dimensional flow: linear and nonlinear dynamics *J. Fluid Mech.* **260** 95–126

[21] Jikov V V, Kozlov S M and Oleinik O A 1994 *Homogenization of Differential Operators and Integral Functionals* (Berlin: Springer)

[22] Grenier E 1997 Oscillatory perturbations of the Navier–Stokes equations *J. Math. Pure Appl.* **76** 477–98

[23] Hruslov E J 1978 Asymptotic behaviour of the solutions of the second boundary value problem in the case of the refinement of the boundary of the domain *Mat. Sb.* **106** 604–21

[24] Kato T 1995 Perturbation theory for linear operators *Classics in Mathematics* (Berlin: Springer)

[25] Kraichnan R H 1976 Eddy viscosity in two and three dimensions *J. Atmos. Sci.* **33** 1521–36

[26] Ladyzhenskaya O A 1969 *The Mathematical Theory of Viscous Incomprehensible Flow* 2nd edn (New York: Gordon and Breach)

[27] Lanotte A, Noullez A, Vergassola M and Wirth A 1999 Large-scale dynamo produced by negative magnetic eddy diffusivities *Geophys. Astrophys. Fluid Dynam.* **91** 131–46

[28] Lions P L 1996 *Mathematical Topics in Fluid Mechanics. Vol 1: Incompressible Models* (*Oxford Lecture Series in Mathematics and Its Applications* vol 10) (Oxford: Oxford University Press)

[29] Mattingly J C and Sinai Ya G 1999 An elementary proof of the existence and uniqueness theorem for the Navier–Stokes equations *Commun. Contemp. Math.* **1** 497–516

[30] Meshalkin L D and Sinai Ya G 1961 Investigation of the stability of a stationary solution of a system of equations for the plane movement of an incompressible viscous liquid *Appl. Math. Mech.* **24** 1700–5

[31] Monin A S and Ozmidov R V 1985 *Turbulence in Oceans* (Dordrecht: Reidel)

[32] Murat F 1977–1978 *H-convergence, Séminaire d´Analyse Fonctionnelle et Numérique* Université d'Alger (Engl. transl. Murat F and Tartar L 1995 H-convergence topics in the mathematical modelling of composite materials *Progress in Nonlinear Differential Equations and Their Applications* ed A Cherkaev and R Kohn (Boston, MA: Birkäuser))

[33] Nguetseng G 1989 A general convergence result for a functional related to the theory of homogenization *SIAM J. Math. Anal.* **20** 608–23

[34] Nepomnyashchy A A 1976 On the stability of the secondary flow of a viscous fluid in an infinite domain *Appl. Math. Mech.* **40** 886–91

[35] Novikov A and Papanicolaou G 2001 Eddy viscosity of cellular flows *J. Fluid Mech.* **446** 173–98

[36] Rekowski B and Rüdiger G 1998 Differential rotation and meridional flow in the solar convection zone with AKA-effect *Astron. Astrophys.* **335** 679–90

[37] Rüdiger G 1989 *Differential Rotation and Stellar Convection* (Berlin: Akademie)

[38] Sell G R, Foias C and Temam R (ed) 1993 *Turbulence in Fluid Flows. A Dynamical Systems Approach* (*The IMA Volumes in Mathematics and its Applications* vol 55) (New York: Springer)

[39] Sánchez-Palencia E 1980 *Nonhomogeneous Media and Vibration Theory* (*Lecture Notes in Physics* vol 127) (Berlin: Springer)

[40] Schochet S 1994 Fast singular limits of hyperbolic PDEs *J. Diff. Eqns* **114** 476–512

[41] Sipp D and Jacquin L 1998 Elliptic instability in two-dimensional flattened Taylor–Green vortices *Phys. Fluids* **10** 839–49

[42] Sivashinsky G I 1985 Weak turbulence in periodic flows *Physica* D **17** 243–55

[43] Sivashinsky G I and Frenkel A L 1992 On negative eddy viscosity under conditions of isotropy *Phys. Fluids* A **4** 1608–10

[44] Sivashinsky G I and Yakhot V 1985 Negative viscosity effect in large-scale flows *Phys. Fluids* **28** 1040–2

[45] Starr V P 1968 *Physics of Negative Viscosity Phenomena* (New York: McGraw-Hill)

[46] Sulem P L, She Z S, Scholl H and Frisch U 1989 Generation of large-scale structures in three-dimensional flow lacking parity-invariance *J. Fluid Mech.* **205** 341–58

[47] Tartar L 1977 *Problemes d'homogenization dans les equations aux derivees partielle* Cours Peccot, Collège de France

[48] Temam R 1984 *Navier–Stokes Equations. Theory and Numerical Analysis* 3rd edn (Amsterdam: North-Holland)

[49] Temam R 1997 *Infinite-Dimensional Dynamical Systems in Mechanics and Physics* 2nd edn (*Applied Mathematical Sciences* vol 68) (New York: Springer)

[50] Temam R 1995 *Navier–Stokes Equations and Nonlinear Functional Analysis* 2nd edn (*CBMS-NSF Regional Conf. Series in Applied Mathematics* vol 66) (Philadelphia, PA: SIAM)

[51] Yudovich V I 1965 Example of secondary stationary flow or periodic flow appearing while a laminar flow of a viscous incompressible fluid loses its stability *Appl. Math. Mech.* **29** 453–67

[52] Yudovich V I 1989 *The Linearization Method in Hydrodynamical Stability Theory* (*Translations of Mathematical Monographs* vol 74) (Providence, RI: AMS)

[53] Zheligovsky V A, Podvigina O M and Frisch U 2001 Dynamo effect in parity-invariant flow with large and moderate separation of scales *Geophys. Astrophys. Fluid Dynam.* **95** 227–68